CollegeBoard Pre-AP®

Pre-AP® Algebra 2

COURSE GUIDE

INCLUDES

- Approach to teaching and learning
- Course map
- Course framework
- Sample assessment questions

Pre-AP® Algebra 2

COURSE GUIDE

Please visit Pre-AP online at **preap.org** for more information and updates about the course and program features.

ABOUT COLLEGE BOARD

College Board is a mission-driven not-for-profit organization that connects students to college success and opportunity. Founded in 1900, College Board was created to expand access to higher education. Today, the membership association is made up of over 6,000 of the world's leading educational institutions and is dedicated to promoting excellence and equity in education. Each year, College Board helps more than seven million students prepare for a successful transition to college through programs and services in college readiness and college success—including the SAT® and The Advanced Placement® Program (AP®). The organization also serves the education community through research and advocacy on behalf of students, educators, and schools.

For further information, visit **www.collegeboard.org**.

PRE-AP EQUITY AND ACCESS POLICY

College Board believes that all students deserve engaging, relevant, and challenging grade-level coursework. Access to this type of coursework increases opportunities for all students, including groups that have been traditionally underrepresented in AP and college classrooms. Therefore, the Pre-AP Program is dedicated to collaborating with educators across the country to ensure all students have the supports to succeed in appropriately challenging classroom experiences that allow students to learn and grow. It is only through a sustained commitment to equitable preparation, access, and support that true excellence can be achieved for all students, and the Pre-AP Course Designation requires this commitment.

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About Pre-AP

Introduction to Pre-AP

Every student deserves classroom opportunities to learn, grow, and succeed. College Board developed Pre-AP® to deliver on this simple premise. Pre-AP courses are designed to support all students across varying levels of readiness. They are not honors or advanced courses.

Participation in Pre-AP courses allows students to slow down and focus on the most essential and relevant concepts and skills. Students have frequent opportunities to engage deeply with texts, sources, and data as well as compelling higher-order questions and problems. Across Pre-AP courses, students experience shared instructional practices and routines that help them develop and strengthen the important critical thinking skills they will need to employ in high school, college, and life. Students and teachers can see progress and opportunities for growth through varied classroom assessments that provide clear and meaningful feedback at key checkpoints throughout each course.

DEVELOPING THE PRE-AP COURSES

Pre-AP courses are carefully developed in partnership with experienced educators, including middle school, high school, and college faculty. Pre-AP educator committees work closely with College Board to ensure that the course resources define, illustrate, and measure grade-level-appropriate learning in a clear, accessible, and engaging way. College Board also gathers feedback from a variety of stakeholders, including Pre-AP partner schools from across the nation who have participated in multiyear pilots of select courses. Data and feedback from partner schools, educator committees, and advisory panels are carefully considered to ensure that Pre-AP courses provide all students with grade-level-appropriate learning experiences that place them on a path to college and career readiness.

PRE-AP PROGRAM COMMITMENTS

The Pre-AP Program asks participating schools to make four commitments:

1. **Pre-AP for All:** Pre-AP frameworks and assessments serve as the foundation for all sections of the course at the school.
2. **Course Frameworks:** Teachers align their classroom instruction with the Pre-AP course frameworks.
   - Schools commit to provide the core resources to ensure Pre-AP teachers and students have the materials they need to engage in the course.

3. **Assessments:** Teachers administer at least one learning checkpoint per unit, on Pre-AP Classroom, and four performance tasks.
4. **Professional Learning:** Teachers complete the foundational professional learning (Online Foundational Modules or Pre-AP Summer Institute) and at least one online performance task scoring module. The current Pre-AP coordinator completes the Pre-AP Coordinator Online Module.

PRE-AP EDUCATOR NETWORK

Similar to the way in which teachers of Advanced Placement® (AP®) courses can become more deeply involved in the program by becoming AP Readers or workshop consultants, Pre-AP teachers also have opportunities to become active in their educator network. Each year, College Board expands and strengthens the Pre-AP National Faculty—the team of educators who facilitate Pre-AP Professional Learning Workshops. Pre-AP teachers can also become curriculum and assessment contributors by working with College Board to design, review, or pilot the course resources.

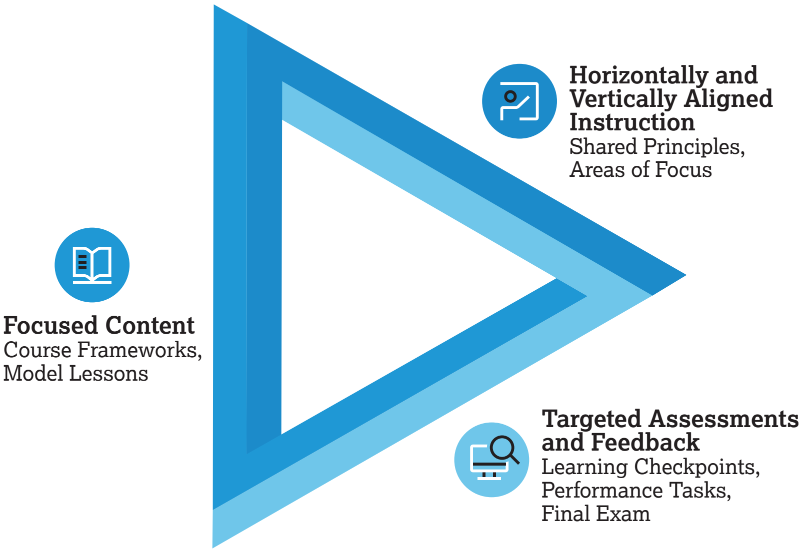
HOW TO GET INVOLVED

Schools and districts interested in learning more about participating in Pre-AP should visit **preap.org/join** or contact us at **preap@collegeboard.org**.

Teachers interested in becoming members of Pre-AP National Faculty or participating in content development should visit **preap.org/national-faculty** or contact us at **preap@collegeboard.org**.

Pre-AP Approach to Teaching and Learning

Pre-AP courses invite all students to learn, grow, and succeed through focused content, horizontally and vertically aligned instruction, and targeted assessments for learning. The Pre-AP approach to teaching and learning, as described below, is not overly complex, yet the combined strength results in powerful and lasting benefits for both teachers and students. This is our theory of action.



FOCUSED CONTENT

Pre-AP courses focus deeply on a limited number of concepts and skills with the broadest relevance for high school coursework and college and career success. The course framework serves as the foundation of the course and defines these prioritized concepts and skills. Pre-AP model lessons and assessments are based directly on this focused framework. The course design provides students and teachers with intentional permission to slow down and focus.

HORIZONTALLY AND VERTICALLY ALIGNED INSTRUCTION

Shared principles cut across all Pre-AP courses and disciplines. Each course is also aligned to discipline-specific areas of focus that prioritize the critical reasoning skills and practices central to that discipline.

SHARED PRINCIPLES

All Pre-AP courses share the following set of research-supported instructional principles. Classrooms that regularly focus on these cross-disciplinary principles allow students to effectively extend their content knowledge while strengthening their critical thinking skills. When students are enrolled in multiple Pre-AP courses, the horizontal alignment of the shared principles provides students and teachers across disciplines with a shared language for their learning and investigation, and multiple opportunities to practice and grow. The critical reasoning and problem-solving tools students develop through these shared principles are highly valued in college coursework and in the workplace.

Close Observation and Analysis

Students are provided time to carefully observe one data set, text image, performance piece, or problem before being asked to explain, analyze, or evaluate. This creates a safe entry point to simply express what they notice and what they wonder. It also encourages students to slow down and capture relevant details with intentionality to support more meaningful analysis, rather than rush to completion at the expense of understanding.

Higher-Order Questioning

Students engage with questions designed to encourage thinking that is elevated beyond simple memorization and recall. Higher-order questions require students to make predictions, synthesize, evaluate, and compare. As students grapple with these questions, they learn that being inquisitive promotes extended thinking and leads to deeper understanding.

Evidence-Based Writing

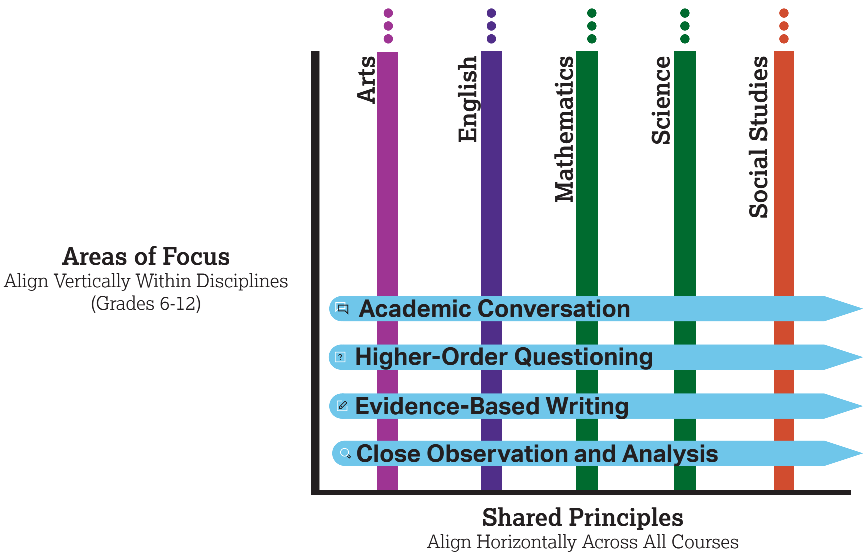
With strategic support, students frequently engage in writing coherent arguments from relevant and valid sources of evidence. Pre-AP courses embrace a purposeful and scaffolded approach to writing that begins with a focus on precise and effective sentences before progressing to longer forms of writing.

Academic Conversation

Through peer-to-peer dialogue, students' ideas are explored, challenged, and refined. As students engage in academic conversation, they come to see the value in being open to new ideas and modifying their own ideas based on new information. Students grow as they frequently practice this type of respectful dialogue and critique and learn to recognize that all voices, including their own, deserve to be heard.

AREAS OF FOCUS

The areas of focus are discipline-specific reasoning skills that students develop and leverage as they engage with content. Whereas the shared principles promote horizontal alignment across disciplines, the areas of focus provide vertical alignment within a discipline, giving students the opportunity to strengthen and deepen their work with these skills in subsequent courses in the same discipline.



For information about the Pre-AP mathematics areas of focus, see page 13.

TARGETED ASSESSMENTS FOR LEARNING

Pre-AP courses include strategically designed classroom assessments that serve as tools for understanding progress and identifying areas that need more support. The assessments provide frequent and meaningful feedback for both teachers and students across each unit of the course and for the course as a whole. For more information about assessments in Pre-AP Algebra 2, see page 53.

Pre-AP Professional Learning

As part of the program commitments, Pre-AP teachers agree to engage in two professional learning opportunities:

1. The first commitment is designed to help prepare teachers to teach their specific course. There are two options to meet this commitment: the Pre-AP Summer Institute (Pre-APSI) and the Online Foundational Modules. Both options provide continuing education units upon completion.
   - The Pre-AP Summer Institute provides a collaborative experience that empowers participants to prepare and plan for their Pre-AP course. While attending, teachers engage with Pre-AP course frameworks, shared principles, areas of focus, and sample model lessons. Participants are given supportive planning time where they work with peers to begin building their Pre-AP course plan.
   - Online Foundational Modules are available to all teachers of Pre-AP courses. In their 12- to 20-hour asynchronous course, teachers explore course materials and experience model lessons from the student's point of view. They also begin building their Pre-AP course plan.
2. The second professional learning opportunity helps teachers prepare for the performance tasks. As part of this commitment, teachers agree to complete at least one online performance task scoring module. Online scoring modules offer guidance and practice applying scoring guidelines and examining student work. Teachers may complete the modules independently or with teachers of the same course in their school's professional learning communities.

About Pre-AP Algebra 2

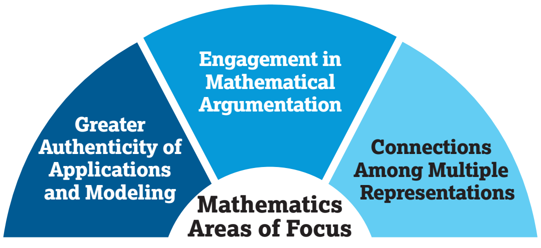
Introduction to Pre-AP Algebra 2

Pre-AP Algebra 2 is designed to optimize students' readiness for college-level mathematics classes. Rather than seeking to cover all topics traditionally included in a standard second-year algebra textbook, this course extends the conceptual understanding of and procedural fluency with functions and data analysis that students developed in their previous mathematics courses. It offers an approach that concentrates on the mathematical content and skills that matter most for college readiness. This approach creates more equitable opportunities for students to take AP STEM courses, especially for those students who are underrepresented in STEM courses and careers. The Pre-AP Algebra 2 Course Framework highlights how to guide students to connect core ideas within and across the units of the course, promoting a coherent understanding of functions.

The components of this course have been crafted to prepare not only the next generation of mathematicians, scientists, programmers, statisticians, and engineers, but also a broader base of mathematically informed citizens who are well equipped to respond to the array of mathematics-related issues that impact our lives at the personal, local, and global levels.

PRE-AP MATHEMATICS AREAS OF FOCUS

The Pre-AP mathematics areas of focus, shown below, are mathematical practices that students develop and leverage as they engage with content. They were identified through educator feedback and research about where students and teachers need the most curriculum support. These areas of focus are vertically aligned to the mathematical practices embedded in other mathematics courses in high school, including AP, and in college, giving students multiple opportunities to strengthen and deepen their work with these skills throughout their educational career. They also support and align to the AP Calculus Mathematical Practices, the AP Statistics Course Skills, and the mathematical practices listed in various state standards.



Greater Authenticity of Applications and Modeling

Students create and use mathematical models to understand and explain authentic scenarios.

Mathematical modeling is a process that helps people analyze and explain the world. In Pre-AP Algebra 2, students explore real-world contexts where mathematics can be used to make sense of a situation. They engage in the modeling process by making choices about what function to use to construct a model, assessing how well the model represents the available data, refining their model as needed, drawing conclusions from their model, and justifying decisions they make through the process.

In addition to mathematical modeling, students engage in mathematics through authentic applications. Applications are similar to modeling problems in that they are drawn from real-world phenomena, but they differ because the applications dictate the appropriate mathematics to use to solve the problem. Pre-AP Algebra 2 balances these two types of real-world tasks.

Engagement in Mathematical Argumentation

Students use evidence to craft mathematical conjectures and prove or disprove them.

Conjecture, reasoning, and proof lie at the heart of the discipline of mathematics. Mathematics is both a way of thinking and a set of tools for solving problems. Pre-AP Algebra 2 students gain proficiency in constructing arguments with definitions of mathematical concepts, reasoning to solve equations, developing skills in using algebra to make sense of data, and crafting assertions using data as evidence and support. Through mathematical argumentation, students learn how to be critical of their own reasoning and the reasoning of others.

Connections Among Multiple Representations

Students represent mathematical concepts in a variety of forms and move fluently among the forms.

Pre-AP Algebra 2 students explore how to weave together multiple representations of function concepts. Every mathematical representation illuminates certain characteristics of a concept while also obscuring other aspects. Throughout the course, students continue to represent mathematical concepts using a variety of forms, allowing them to develop a nuanced understanding of which representations best serve a particular purpose.

PRE-AP ALGEBRA 2 AND CAREER READINESS

The Pre-AP Algebra 2 course resources are designed to expose students to a wide range of career opportunities that depend on algebraic knowledge and skills. Examples include not only field-specific specialty careers such as mathematician and statistician, but also other endeavors where algebraic knowledge is relevant, such as accounting, economics, engineering, and programming.

Career clusters that involve algebra, along with examples of careers in mathematics or related to mathematics, are provided below. Teachers should consider discussing these with students throughout the year to promote motivation and engagement.

Career Clusters Involving Mathematics	
architecture and construction arts, A/V technology, and communications business management and administration finance government and public administration health science information technology marketing STEM (science, technology, engineering, and math) transportation, distribution, and logistics	
Examples of Mathematics Related Careers	**Examples of Algebra 2 Related Careers**
actuary financial analyst mathematician mathematics teacher professor programmer statistician	accountant computer programmer economist electrician engineer health science technician operations research analyst

Source for Career Clusters: "Advanced Placement and Career and Technical Education: Working Together." Advance CTE and the College Board. October 2018. **https://careertech.org/resource/ap-cte-working-together**.

For more information about careers that involve mathematics, teachers and students can visit and explore the College Board's Big Future resources: **https://bigfuture.collegeboard.org/majors/math-statistics-mathematics**.

SUMMARY OF RESOURCES AND SUPPORTS

Teachers are strongly encouraged to take advantage of the full set of resources and supports for Pre-AP Algebra 2, which is summarized below. Some of these resources are part of the Pre-AP Program commitments that lead to Pre-AP Course Designation. To learn more about the commitments for course designation, see details below and on page 75.

COURSE FRAMEWORK

Included in this guide as well as in the *Pre-AP Algebra 2 Teacher Resources*, the course framework defines what students should know and be able to do by the end of the course. It serves as an anchor for model lessons and assessments, and it is the primary resource needed to plan the course. **Teachers commit to aligning their classroom instruction with the course framework**. *For more details see page 20.*

MODEL LESSONS

Teacher resources, available in print and online, include a robust set of model lessons that demonstrate how to translate the course framework, shared principles, and areas of focus into daily instruction. **Use of the model lessons is encouraged**. *For more details see page 51.*

LEARNING CHECKPOINTS

Accessed through Pre-AP Classroom, these short formative assessments provide insight into student progress. They are automatically scored and include multiple-choice and technology-enhanced items with rationales that explain correct and incorrect answers. **Teachers commit to administering one learning checkpoint per unit**. *For more details see page 53.*

PERFORMANCE TASKS

Available in the printed teacher resources as well as on Pre-AP Classroom, performance tasks allow students to demonstrate their learning through extended problem-solving, writing, analysis, and/or reasoning tasks. Scoring guidelines are provided to inform teacher scoring, with additional practice and feedback suggestions available in online modules on Pre-AP Classroom. **Teachers commit to using each unit's performance task**. *For more details see page 55.*

PRACTICE PERFORMANCE TASKS

Available in the student resources, with supporting materials in the teacher resources, these tasks provide an opportunity for students to practice applying skills and knowledge as they would in a performance task, but in a more scaffolded environment. **Use of the practice performance tasks is encouraged**. *For more details see page 55.*

FINAL EXAM

Accessed through Pre-AP Classroom, the final exam serves as a classroom-based, summative assessment designed to measure students' success in learning and applying the knowledge and skills articulated in the course framework. **Administration of the final exam is encouraged**. *For more details see page 69.*

PROFESSIONAL LEARNING

Both the Pre-AP Summer Institute (Pre-APSI) and the Online Foundational Modules support teachers in preparing and planning to teach their Pre-AP course. **All Pre-AP teachers make a commitment to either attend the Pre-APSI (in person or virtually) or complete the Online Foundational Modules. In addition, teachers agree to complete at least one Online Performance Task Scoring module**. *For more details see page 9.*

Course Map

PLAN

The course map shows how components are positioned throughout the course. As the map indicates, the course is designed to be taught over 140 class periods (based on 45-minute class periods), for a total of 28 weeks.

Model lessons are included for approximately 50% of the total instructional time, with the percentage varying by unit. Each unit is divided into key concepts.

TEACH

The model lessons demonstrate how the Pre-AP shared principles and mathematics areas of focus come to life in the classroom.

Shared Principles
Close observation and analysis
Higher-order questioning
Evidence-based writing
Academic conversation

Areas of Focus
Greater authenticity of applications and modeling
Engagement in mathematical argumentation
Connections among multiple representations

ASSESS AND REFLECT

Each unit includes two learning checkpoints and a performance task. These formative assessments are designed to provide meaningful feedback for both teachers and students.

Note: The final exam, offered during a six-week window in the spring, is not represented on the map.

UNIT 1 Modeling with Functions

~35 Class Periods

Pre-AP model lessons provided for 40% of instructional time in this unit

KEY CONCEPT 1.1
Choosing Appropriate Function Models

Learning Checkpoint 1

KEY CONCEPT 1.2
Rate of Change

Performance Task for Unit 1

KEY CONCEPT 1.3
Piecewise-Defined Models

Learning Checkpoint 2

UNIT 2 The Algebra of Functions

~30 Class Periods

Pre-AP model lessons provided for approximately 40% of instructional time in this unit

KEY CONCEPT 2.1
Composing Functions

KEY CONCEPT 2.2
Transforming Functions

Learning Checkpoint 1

KEY CONCEPT 2.3
Inverting Functions

Learning Checkpoint 2

Performance Task for Unit 2

UNIT 3 Function Families

~45 Class Periods

Pre-AP model lessons provided for approximately 20% of instructional time in this unit

KEY CONCEPT 3.1
Exponential and Logarithmic Functions

Learning Checkpoint 1

KEY CONCEPT 3.2
Polynomial and Rational Functions

Performance Task for Unit 3

KEY CONCEPT 3.3
Square Root and Cube Root Functions

Learning Checkpoint 2

UNIT 4T Trigonometric Functions

~30 Class Periods

Pre-AP model lessons provided for approximately 40% of instructional time in this unit

KEY CONCEPT 4T.1
Radian Measure and Sinusoidal Functions

Performance Task for Unit 4T

Learning Checkpoint 1

KEY CONCEPT 4T.2
The Tangent Function and Other Trigonometric Functions

KEY CONCEPT 4T.3
Inverting Trigonometric Functions

Learning Checkpoint 2

UNIT 4M Matrices and Their Applications

~30 Class Periods

Pre-AP model lessons provided for approximately 40% of instructional time in this unit

KEY CONCEPT 4M.1
Geometric Transformations

Learning Checkpoint 1

KEY CONCEPT 4M.2
Solving Systems of Equations with Matrices

KEY CONCEPT 4M.3
Applications of Matrix Multiplication

Learning Checkpoint 2

Performance Task for Unit 4M

Note: Schools can choose to complete either Unit 4T **or** Unit 4M, depending on which unit is the best fit for state or local standards.

Pre-AP Algebra 2 Course Framework

INTRODUCTION

Based on the Understanding by Design® (Wiggins and McTighe) model, the Pre-AP Algebra 2 Course Framework is back mapped from AP expectations and aligned to essential grade-level expectations. The course framework serves as a teacher's blueprint for the Pre-AP Algebra 2 instructional resources and assessments.

The course framework was designed to meet the following criteria:

- **Focused:** The framework provides a deep focus on a limited number of concepts and skills that have the broadest relevance for later high school, college, and career success.
- **Measurable:** The framework's learning objectives are observable and measurable statements about the knowledge and skills students should develop in the course.
- **Manageable:** The framework is manageable for a full year of instruction, fosters the ability to explore concepts in depth, and enables room for additional local or state standards to be addressed where appropriate.
- **Accessible:** The framework's learning objectives are designed to provide all students, across varying levels of readiness, with opportunities to learn, grow, and succeed.

COURSE FRAMEWORK COMPONENTS

The Pre-AP Algebra 2 Course Framework includes the following components:

Big Ideas

The big ideas are recurring themes that allow students to create meaningful connections between course concepts. Revisiting the big ideas throughout the course and applying them in a variety of contexts allow students to develop deeper conceptual understandings.

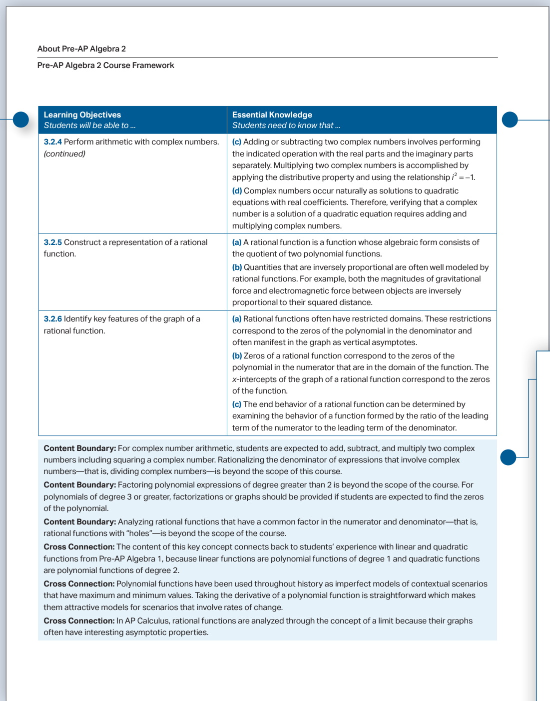
Enduring Understandings

Each unit focuses on a small set of enduring understandings. These are the long-term takeaways related to the big ideas that leave a lasting impression on students. Students build these understandings over time by exploring and applying course content throughout the year.

Key Concepts

To support teacher planning and instruction, each unit is organized by key concepts. Each key concept includes relevant **learning objectives** and **essential knowledge statements** and may also include **content boundary and cross connection statements**. These are illustrated and defined below.

Learning Objectives:
These objectives define what a student needs to be able to do with essential knowledge to progress toward the enduring understandings. The learning objectives serve as actionable targets for instruction and assessment.



About Pre-AP Algebra 2

Pre-AP Algebra 2 Course Framework

Learning Objectives *Students will be able to ...*	Essential Knowledge *Students need to know that ...*
3.2.4 Perform arithmetic with complex numbers. *(continued)*	**(c)** Adding or subtracting two complex numbers involves performing the indicated operation with the real parts and the imaginary parts separately. Multiplying two complex numbers is accomplished by applying the distributive property and using the relationship $i^2 = -1$. **(d)** Complex numbers occur naturally as solutions to quadratic equations with real coefficients. Therefore, verifying that a complex number is a solution of a quadratic equation requires adding and multiplying complex numbers.
3.2.5 Construct a representation of a rational function.	**(a)** A rational function is a function whose algebraic form consists of the quotient of two polynomial functions. **(b)** Quantities that are inversely proportional are often well modeled by rational functions. For example, both the magnitudes of gravitational force and electromagnetic force between objects are inversely proportional to their squared distance.
3.2.6 Identify key features of the graph of a rational function.	**(a)** Rational functions often have restricted domains. These restrictions correspond to the zeros of the polynomial in the denominator and often manifest in the graph as vertical asymptotes. **(b)** Zeros of a rational function correspond to the zeros of the polynomial in the numerator that are in the domain of the function. The x-intercepts of the graph of a rational function correspond to the zeros of the function. **(c)** The end behavior of a rational function can be determined by examining the behavior of a function formed by the ratio of the leading term of the numerator to the leading term of the denominator.

Content Boundary: For complex number arithmetic, students are expected to add, subtract, and multiply two complex numbers including squaring a complex number. Rationalizing the denominator of expressions that involve complex numbers—that is, dividing complex numbers—is beyond the scope of this course.

Content Boundary: Factoring polynomial expressions of degree greater than 2 is beyond the scope of the course. For polynomials of degree 3 or greater, factorizations or graphs should be provided if students are expected to find the zeros of the polynomial.

Content Boundary: Analyzing rational functions that have a common factor in the numerator and denominator—that is, rational functions with "holes"—is beyond the scope of the course.

Cross Connection: The content of this key concept connects back to students' experience with linear and quadratic functions from Pre-AP Algebra 1, because linear functions are polynomial functions of degree 1 and quadratic functions are polynomial functions of degree 2.

Cross Connection: Polynomial functions have been used throughout history as imperfect models of contextual scenarios that have maximum and minimum values. Taking the derivative of a polynomial function is straightforward which makes them attractive models for scenarios that involve rates of change.

Cross Connection: In AP Calculus, rational functions are analyzed through the concept of a limit because their graphs often have interesting asymptotic properties.

Essential Knowledge Statements:
Each essential knowledge statement is linked to a learning objective. One or more essential knowledge statements describe the knowledge required to perform each learning objective.

Content Boundary and Cross Connection Statements:
When needed, content boundary statements provide additional clarity about the content and skills that lie within versus beyond the scope of this course.

Cross connection statements highlight important connections that should be made between key concepts within and across the units.

BIG IDEAS IN PRE-AP ALGEBRA 2

While the Pre-AP Algebra 2 framework is organized into four core units of study, the content is grounded in three big ideas, which are cross-cutting concepts that build conceptual understanding and spiral throughout the course. Since these ideas cut across units, they serve as the underlying foundation for the enduring understandings, key concepts, and learning objectives that make up the focus of each unit. A deep and productive understanding in Pre-AP Algebra 2 relies on these three big ideas:

- **Function:** The mathematical concept of function describes a special kind of relationship where each input value corresponds to a single output value. Functions are among the most important objects in modern mathematics. Functions can be constructed to model phenomena that involve quantities such as time, force, and money, among others. These function models of real-world phenomena allow us to discover and investigate patterns among the related quantities. Studying patterns through the lens of a function model provides insights that lead to reasoned predictions and sound decision making.
- **Operations with Functions:** Two functions can be combined, composed, and transformed to form a new function that is a better model of a real-world phenomenon than the original function. Operations on functions include the arithmetic operations of addition, subtraction, multiplication, and division, as well as a special kind of operation, composition. Composition is the process of using the output of one function as the input of another function. When one of the functions in the composition is either a sum or product of a constant and a variable, the composition is referred to as a function transformation because the effect of such an operation on the graph of the function can be described in terms of geometric transformations. A thorough understanding of how to use function operations to construct more complex and nuanced function models is critical to the success of the mathematical modeling process.
- **Inverse Functions:** Solving an equation often relies on undoing an operation with its inverse operation. The processes of inverse operations are formalized as the concept of an inverse function, which expresses the idea that some mathematical relationships can be reversed. If a function defines a way to determine the output value from an input value, an inverse function defines a way to determine the input value from an output value. This implies that patterns observed in the output of one function can be seen in the input of the inverse function. Some of the functions covered in the course, such as exponential and logarithmic functions, are inverses of each other.

OVERVIEW OF PRE-AP ALGEBRA 2 UNITS AND ENDURING UNDERSTANDINGS

Unit 1: Modeling with Functions	Unit 2: The Algebra of Functions
▪ Many bivariate data sets can be appropriately modeled by linear, quadratic, or exponential functions because the relationships between the quantities exhibit characteristics similar to those functions. ▪ Mathematical functions almost never perfectly fit a real-world context, but a function model can be useful for making sense of that context. ▪ Average rate of change allows us to understand multifaceted relationships between quantities by modeling them with linear functions.	▪ Composing functions allows simpler functions to be combined to construct a function model that more appropriately captures the characteristics of a contextual scenario. ▪ Transformations are a special kind of composition. When one of the functions being composed consists only of addition or multiplication, the effects on the other function are straightforward to determine. ▪ An inverse function defines the way to determine the input value that corresponds to a given output value.
Unit 3: Function Families	**Unit 4T: Trigonometric Functions**
▪ A function is a special mathematical relationship between two variables that can often be used to make sense of observable patterns in contextual scenarios. ▪ Functions in a family have similar properties, similar algebraic representations, and graphs that share key features.	▪ Trigonometry connects the study of circles and the study of right triangles. ▪ Real-world contexts that exhibit periodic behavior or circular motion can be modeled by trigonometric functions.
Unit 4M: Matrices and Their Applications	
▪ A matrix is a tool that can be used to efficiently organize mathematical information. ▪ Matrix multiplication provides efficient ways to evaluate multiple linear expressions simultaneously. ▪ Systems of linear equations can be solved using inverse matrices.	

Unit 1: Modeling with Functions

Suggested timing: Approximately 7 weeks

In the first unit of the course, students build upon their previous experience with linear, quadratic, and exponential functions. These important functions form the foundation upon which other functions introduced in this course are built. Unit 1 focuses on using functions to model real-world data sets and contextual scenarios. This focus on modeling provides authentic opportunities for students to investigate and confirm the defining characteristics of linear, quadratic, and exponential functions while simultaneously reinforcing procedural fluency with these function families.

Throughout Pre-AP Algebra 2, students are expected to take ownership of the mathematics they use by crafting arguments for why one type of function is better than another for modeling a particular data set or contextual scenario. This allows students to develop a deeper understanding of these foundational functions as they drive the mathematical modeling process themselves. This requires a more thorough understanding of modeling than prior Pre-AP mathematics courses, in which students were asked to explain why a given function type was an appropriate model for a given data set or contextual scenario.

ENDURING UNDERSTANDINGS

Students will understand that …

- Many bivariate data sets can be appropriately modeled by linear, quadratic, or exponential functions because the relationships between the quantities exhibit characteristics similar to those functions.
- Mathematical functions almost never perfectly fit a real-world context, but a function model can be useful for making sense of that context.
- Average rate of change allows us to understand multifaceted relationships between quantities by modeling them with linear functions.

KEY CONCEPTS

- 1.1: Choosing Appropriate Function Models – Using linear, quadratic, and exponential functions to make sense of relationships between two quantities
- 1.2: Rate of Change – Using linear functions to make sense of complex relationships
- 1.3: Piecewise-Defined Models – Using functions defined over discrete intervals to make sense of contexts with varied characteristics

KEY CONCEPT 1.1: CHOOSING APPROPRIATE FUNCTION MODELS

Using linear, quadratic, and exponential functions to make sense of relationships between two quantities

Learning Objectives *Students will be able to ...*	**Essential Knowledge** *Students need to know that ...*
1.1.1 Identify a function family that would appropriately model a data set or contextual scenario.	**1.1.1a** Linear functions often appropriately model data sets that exhibit a roughly constant rate of change. **1.1.1b** Quadratic functions often appropriately model data sets that exhibit roughly linear rates of change, are roughly symmetric, and have a unique maximum or minimum output value. **1.1.1c** Exponential functions often appropriately model data sets that exhibit roughly constant ratios of output values over equal intervals of input values.
1.1.2 Use residual plots to determine whether a function model appropriately models a data set.	**1.1.2a** The residual for a data point is the deviation between the observed data value and the value predicted by a function model. Graphically, this can be thought of as the vertical segment between the data point and the graph of the function model. **1.1.2b** A residual plot is a scatterplot of values of the independent variable and their corresponding residuals. **1.1.2c** The sign of the residual indicates whether the function model is an overestimate or underestimate of the observed data value. **1.1.2d** An appropriate function model for a data set produces a residual plot with no discernible pattern. Residual plots that display some systematic pattern indicate that there is variation in the data not accounted for in the function model.
1.1.3 Construct a representation of a linear, quadratic, or exponential function both with and without technology.	**1.1.3a** A linear function can be expressed in slope-intercept form to reveal the constant rate of change and the initial value, or in point-slope form to reveal the constant rate of change and one ordered pair that satisfies the relationship. **1.1.3b** A quadratic function can be expressed in vertex form to reveal its maximum or minimum value; in factored form to reveal the zeros of the function, which often correspond to the boundaries of the contextual domain; or in standard form to reveal the initial value. **1.1.3c** An exponential function can be expressed in the form $f(x)=a(1+r)^x$ to reveal the percent change in the output, r, for a one-unit change in the input, or in the form $f(x)=a \bullet b^{\left(\frac{x}{n}\right)}$ to reveal the growth/decay factor, b, over an n-unit change in the input. **1.1.3d** A function within a function family that best fits a data set minimizes the error of the function model, which is often quantified by the sum of the squares of the residuals.

Learning Objectives *Students will be able to ...*	**Essential Knowledge** *Students need to know that ...*
1.1.4 Use a function that models a data set or contextual scenario to predict values of the dependent variable.	**1.1.4a** An appropriate model for a bivariate data set can be used to predict values of the dependent variable from a value of the independent variable. **1.1.4b** Functions that model a data set are frequently only useful over their contextual domain.

Content Boundary: A primary focus for this key concept is the use of functions as models for data sets and contextual scenarios. Calculating a function model from a large data set by hand is beyond the scope of the course. The use of technology to determine a function model for a data set is strongly encouraged; however, the analysis arising from a function model is best done by the student.

Cross Connection: In this key concept, students build upon their experience with scatterplots and trend lines from Pre-AP Algebra 1. Through this unit, they see that some data sets are best modeled by linear functions, while other data sets are more appropriately modeled by quadratic or exponential functions.

Cross Connection: Because linear, quadratic, and exponential functions are the most broadly useful functions for making sense of real-world phenomena, developing deep conceptual understanding and procedural fluency supports student success on SAT and access to advanced mathematics courses.

KEY CONCEPT 1.2: RATE OF CHANGE

Using linear functions to make sense of complex relationships

Learning Objectives *Students will be able to …*	**Essential Knowledge** *Students need to know that …*
1.2.1 Interpret the average rate of change of a function over a given interval, including contextual scenarios.	**1.2.1a** The average rate of change of a function over an interval can be interpreted as the constant rate of change of a linear function with the same change in output for the given change in the input. This constant rate of change can be interpreted graphically as the slope of the line between the points on the ends of the interval. **1.2.1b** The average rate of change of a function f over the interval $[a, b]$ is the ratio $\frac{f(b)-f(a)}{b-a}$. That is, the average rate of change is $\frac{\Delta f(x)}{\Delta x}$.
1.2.2 Predict values of a function using the average rate of change and an input-output pair of a function model.	**1.2.2a** The average rate of change of a function over the interval $[a, b]$ can be used to estimate values of the function within or near the interval. **1.2.2b** The change in the value of $f(x)$ over an interval of width Δx can be determined by the product of the average rate of change of f and Δx.

Cross Connection: This key concept builds directly on students' understanding from Pre-AP Algebra 1 that linear functions have a constant rate of change. That prior knowledge can be leveraged here as students come to see how linear functions are used to make sense of more complex scenarios.

Cross Connection: The concept of average rate of change over increasingly smaller intervals is the basis for understanding the derivative of a function. In a calculus class, students will learn that the average rate of change of a function f over the interval $[x, x+\Delta x]$ is determined by the difference quotient, $\frac{f(x+\Delta x)-f(x)}{\Delta x}$.

KEY CONCEPT 1.3: PIECEWISE-DEFINED MODELS

Using functions defined over discrete intervals to make sense of contexts with varied characteristics

Learning Objectives *Students will be able to ...*	**Essential Knowledge** *Students need to know that ...*
1.3.1 Construct a representation of a piecewise-defined function.	**1.3.1a** A piecewise-defined function is a function that is defined on a set of nonoverlapping intervals. **1.3.1b** Data sets or contextual scenarios that demonstrate different characteristics, such as rates of change, over different intervals of the domain would be appropriately modeled by a piecewise-defined function. **1.3.1c** An algebraic representation of a piecewise-defined function consists of multiple algebraic expressions that describe the function over nonoverlapping intervals of the domain. **1.3.1d** The graph of a piecewise-defined function is the set of input-output coordinate pairs that satisfy the function relationship.
1.3.2 Evaluate a piecewise-defined function at specified values of the domain.	**1.3.2a** The output value of a piecewise-defined function for a specific input is determined by applying the algebraic rule for which the input value is defined. **1.3.2b** Output values of a piecewise-defined function can be estimated, and sometimes determined, from a graph of the function.
1.3.3 Construct a representation of an absolute value function.	**1.3.3a** The absolute value function is algebraically denoted as $f(x)=\|x\|$. **1.3.3b** The function $f(x)=\|x\|$ is a piecewise-defined function. If x is nonnegative, then $\|x\|=x$; if x is negative, then $\|x\|=-x$. The graph of $y=f(x)$ consists of $y=x$ for $x \geq 0$ and $y=-x$ for $x<0$.

Content Boundary: Intervals of real numbers can be expressed in interval notation or in inequality notation. Students are expected to be familiar with reading and writing intervals using both notations. The particular context in which an interval of real numbers is used should determine which notation is more appropriate.

Unit 2: The Algebra of Functions

Suggested timing: Approximately 6 weeks

In Unit 2, students develop a conceptual understanding of the algebra of functions and build procedural fluency with function notation. Students tend to think about transformations of functions and composition of functions as unrelated topics. In this unit, students connect these important concepts to develop a more coherent understanding of functions by first exploring function composition, a new operation that chains functions together in a sequence. Once students understand the power of function composition, they work to see how function transformations are a special case of composition in which a given function is composed with a linear function.

The unit culminates in an exploration of inverses—the mathematical concept of undoing—through inverse operations and inverse functions. Students develop familiarity with inverse operations through their elementary school experiences with addition and multiplication, and their respective inverses, subtraction and division. In this unit, the inverse operation of exponentiating—taking a logarithm—is introduced. From prior coursework, students know that a function associates each input with one output. In this course, students learn that if a function has an inverse function, it associates an output back to its input. By considering inverses as both operations and functions, students develop a deep understanding of this critical concept.

ENDURING UNDERSTANDINGS

Students will understand that …

- Composing functions allows simpler functions to be combined to construct a function model that more appropriately captures the characteristics of a contextual scenario.
- Transformations are a special kind of composition. When one of the functions being composed consists only of addition or multiplication, the effects on the other function are straightforward to determine.
- An inverse function defines the way to determine the input value that corresponds to a given output value.

KEY CONCEPTS

- 2.1: Composing Functions – Chaining functions together in a sequence to construct better function models
- 2.2: Transforming Functions – Exploring how addition and multiplication affect the input or output of a function
- 2.3: Inverting Functions – Making sense of doing and undoing through inverse operations and inverse functions

KEY CONCEPT 2.1: COMPOSING FUNCTIONS

Chaining functions together in a sequence to construct better function models

Learning Objectives *Students will be able to ...*	**Essential Knowledge** *Students need to know that ...*
2.1.1 Determine the output value of the composition of two or more functions for a given input value when the functions have the same or different representations.	**2.1.1a** Composing functions is a process in which the output of one function is used as the input of another function. **2.1.1b** Composing functions is generally not a commutative operation. That is, for most functions, the value of $f(g(x))$ is not equal to the value of $g(f(x))$ for a given value of x.
2.1.2 Construct a representation of a composite function when the functions being composed have the same or different representations.	**2.1.2a** Composing two functions, f and g, results in a new function, called the composite function, that can be notated $f \circ g$ where $f \circ g(x) = f(g(x))$. **2.1.2b** An algebraic representation of $f \circ g$ is constructed by substituting every instance of x in the algebraic representation of $f(x)$ with the algebraic representation of $g(x)$. **2.1.2c** A graphical representation of $f \circ g$ can be constructed from the algebraic representation of $f \circ g$ or approximated by plotting some ordered pairs of the form $(x, f(g(x)))$. **2.1.2d** A numerical representation of $f \circ g$ consists of a subset of the ordered pairs that satisfy the relationship and is constructed by directly calculating values of $f(g(x))$ from values of x that are in the domain of g.
2.1.3 Express a given algebraic representation of a function in an equivalent form as the composition of two or more functions.	**2.1.3a** Any function can be expressed as the composition of two or more functions. One of these functions can be the identity function, $f(x) = x$. **2.1.3b** Algebraic techniques, such as factoring, can be used to express an algebraic representation of a function as a composition of functions.

Cross Connection: Students used function composition in Pre-AP Geometry with Statistics when they used sequences of multiple rigid motion and/or similarity transformations to associate one figure with another. In those experiences, the output of one transformation was treated as the input of another transformation. Students learned that changing the order in which the same transformations are applied to a preimage often yields different images. That understanding is reinforced through this key concept when students learn the importance of the order of composing two or more functions.

Cross Connection: Function composition often appears in problems in which the frame of reference for a function model is specified. For example, the output of a function could be the height as measured from the roof instead of the height as measured from the ground, or the input of a function could be the time that has elapsed since noon instead of the time that has elapsed since midnight. Using function composition to change the frame of reference is a valuable technique with applications across mathematics and science courses.

Cross Connection: The work of Learning Objective 2.1.3, understanding how one function can be written as a composition of two or more functions, is a critical first step in understanding the chain rule in AP Calculus. The chain rule defines how to take the derivative of a composite function.

KEY CONCEPT 2.2: TRANSFORMING FUNCTIONS

Exploring how addition and multiplication affect the input or output of a function

Learning Objectives *Students will be able to …*	**Essential Knowledge** *Students need to know that …*
2.2.1 Compare a function f with an additive transformation of f, that is, $f(x+k)$ or $f(x)+k$.	**2.2.1a** An additive transformation of f is a composition of f with $g(x)=x+k$. That is, $f \circ g(x)=f(x+k)$ and $g \circ f(x)=f(x)+k$. **2.2.1b** The graph of an additive transformation of f is either a vertical or horizontal translation of the graph of f. **2.2.1c** If (a, b) is an input-output pair of f, then $(a-k, b)$ must be an input-output pair of $f(x+k)$. Therefore, the graph of $f(x+k)$ is a horizontal translation of the graph of f by $-k$ units. **2.2.1d** If (a, b) is an input-output pair of f, then $(a, b+k)$ must be an input-output pair of $f(x)+k$. Therefore, the graph of $f(x)+k$ is a vertical translation of the graph of f by k units.
2.2.2 Compare a function f with a multiplicative transformation of f, that is, $f(kx)$ or $k \cdot f(x)$.	**2.2.2a** A multiplicative transformation of f is a composition of f with $g(x)=kx$, where $k \neq 0$. That is, $f \circ g(x)=f(kx)$ and $g \circ f(x)=k \cdot f(x)$. **2.2.2b** The graph of a multiplicative transformation of f is either a vertical or horizontal dilation of the graph of f. When $k<0$, the graph of the multiplicative transformation is also a reflection of the graph of f over one of the axes. **2.2.2c** If (a, b) is an input-output pair of f, then $\left(\frac{1}{k} \cdot a, b\right)$ must be an input-output pair of $f(kx)$. Therefore, the graph of $f(kx)$ is a horizontal dilation of the graph of f by a factor of $\left\lvert\frac{1}{k}\right\rvert$. **2.2.2d** If (a, b) is an input-output pair of f, then (a, kb) must be an input-output pair of $k \cdot f(x)$. Therefore, the graph of $k \cdot f(x)$ is a vertical dilation of the graph by a factor of $\lvert k \rvert$.
2.2.3 Construct a representation of the transformation of a function.	**2.2.3a** A function transformation is a sequence of additive and multiplicative transformations of f. The order in which the transformations are applied matters. **2.2.3b** Changing the reference point for the input or output quantity of a function can be achieved with an additive transformation. **2.2.3c** Converting the unit of measure for an input or output quantity of a function can be achieved with a multiplicative transformation.

Cross Connection: Students used additive and multiplicative transformations in Pre-AP Geometry with Statistics in the form of translations and dilations, respectively. Referring back to related geometric concepts in this key concept will help students make deep connections among the various transformations with which they have worked.

KEY CONCEPT 2.3: INVERTING FUNCTIONS

Making sense of doing and undoing through inverse operations and inverse functions

Learning Objectives *Students will be able to ...*	**Essential Knowledge** *Students need to know that ...*
2.3.1 Determine all input values that correspond to a specified output value given a function model on a specified domain.	**2.3.1a** For algebraic representations of an equation, inverse operations, such as squaring/square rooting and cubing/cube rooting, can be used to determine the input values that correspond to a specified output value. **2.3.1b** For algebraic representations of an equation involving an exponential expression, the inverse operation of exponentiating, called taking a logarithm, can be used to determine the input values that correspond to a specified output value. The solution to the equation $b^x = c$ where b and c are both positive and $b \neq 1$, is expressed by $x = \log_b(c)$. **2.3.1c** For graphical representations of an equation, identifying all ordered pairs that lie on the intersection of the line $y = k$ and the graph of $y = f(x)$ provides all input values that correspond with the output value k.
2.3.2 Express exactly or approximate the value of a logarithmic expression as a rational number.	**2.3.2a** An exact value of a logarithm can be determined using laws of exponents. **2.3.2b** If the logarithmic expression $\log_b(c)$ can be expressed exactly as a rational number, then its value is the rational number, x, that makes the equation $b^x = c$ true. **2.3.2c** If the logarithmic expression $\log_b(c)$ cannot be written exactly as a rational number, then its value can be approximated by a rational number x, for which $b^x \approx c$.
2.3.3 Determine a domain over which the inverse function of a specified function is defined.	**2.3.3a** A function f has an inverse function on a specified domain if each output value of f corresponds to exactly one input value in that domain. **2.3.3b** A function f is called invertible on a specified domain if there exists an inverse function, f^{-1}, such that $f(a) = b$ implies $f^{-1}(b) = a$. **2.3.3c** There are multiple ways to restrict the domain of a function so that the function is invertible. The appropriate domain restrictions for making a function invertible may depend on the context.

Learning Objectives *Students will be able to ...*	**Essential Knowledge** *Students need to know that ...*
2.3.4 Construct a representation of the inverse function given a function that is invertible on its domain.	**2.3.4a** A table of values for the inverse function of f consists of all input-output ordered pairs (b, a) such that (a, b) is an input-output ordered pair of f. **2.3.4b** The graph of the inverse function of f is a reflection of the graph of $y = f(x)$ across the line $y = x$. **2.3.4c** The algebraic representation of the inverse function of $y = f(x)$ is determined using inverse operations to express x in terms of y. **2.3.4d** The domain and range of f^{-1} are the range and domain of f, respectively.
2.3.5 Verify that one function is an inverse of another function using composition.	**2.3.5a** If f is an inverse function of g, then g is an inverse function of f. **2.3.5b** If f is an inverse function of g, then composing f and g in either order will map each input onto itself. **2.3.5c** The function f is the inverse of the function g if and only if their composition in either order is the identity function, x. That is, $f(g(x)) = x$ and $g(f(x)) = x$.

Content Boundary: In this key concept, students are expected to solve quadratic and exponential functions using their associated inverse operations, taking the square root and taking a logarithm. Students are expected to develop an intuition for exact and approximate values of square roots of real numbers and of logarithms of real numbers by understanding their relationship to their respective inverses.

Content Boundary: The problems and exercises that address Learning Objective 2.3.4 are limited to linear functions and quadratic functions, with their domains appropriately constrained. Determining an inverse function for an exponential function is beyond the scope of this unit but will be expected in Unit 3.

Cross Connection: In Pre-AP Algebra 1, students learned that the expression $\sqrt{a}$ is a notation for the number whose square is a. Similarly, in this key concept, students learn that the expression $\log_b(c)$ is a notation for the exponent of b, such that b raised to that exponent has a value of c. That is, the equations $x = \log_b(c)$ and $b^x = c$ convey the same information about the relationship between the numbers b and c.

Cross Connection: The concepts of doing and undoing are central to many mathematical concepts. Inverse operations allow equations to be solved methodically rather than through an inefficient guess-and-check process. The existence of an inverse function shows that a function relationship between two quantities can be associated in two directions: from input to output and from output to input. In AP Calculus, the Fundamental Theorem of Calculus establishes an inverse relationship between differentiating and integrating a function.

Unit 3: Function Families

Suggested timing: Approximately 9 weeks

Explorations of function families are an important component of any Algebra 2 course because they expand the repertoire of functions students can draw upon to model real-world phenomena. Not all phenomena are appropriately modeled by a linear, quadratic, or exponential function. For example, the gravitational force between two objects is inversely proportional to the square of their distance apart. This relationship would be best modeled with a rational function, one of the functions introduced in Unit 3. Throughout this unit, students learn that a parent function and its transformations form a function family. All functions in the same function family share some properties with each other.

Because function families are traditionally taught as isolated topics, students rarely have time to thoroughly investigate which properties of a function family are maintained by transformations and which are not. Therefore, the key concepts in this unit intentionally focus students' thinking on how function families are related in meaningful ways. The structure of the unit is intended to help students construct a network of connections among these function families. As with all explorations of functions throughout Pre-AP, the emphasis is on contextual scenarios that can be effectively modeled by each function family.

ENDURING UNDERSTANDINGS

Students will understand that …

- A function is a special mathematical relationship between two variables that can often be used to make sense of observable patterns in contextual scenarios.
- Functions in a family have similar properties, similar algebraic representations, and graphs that share key features.

KEY CONCEPTS

- 3.1: Exponential and Logarithmic Functions – Using functions to make sense of multiplicative patterns of change
- 3.2: Polynomial and Rational Functions – Using functions to make sense of sums and quotients of powers
- 3.3: Square Root and Cube Root Functions – Using functions to make sense of inverting quadratic and cubic relationships

KEY CONCEPT 3.1: EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Using functions to make sense of multiplicative patterns of change

Learning Objectives *Students will be able to ...*	**Essential Knowledge** *Students need to know that ...*
3.1.1 Construct a representation of an exponential function using the natural base, *e*.	**3.1.1a** The natural base *e*, which is approximately 2.718, is often used as the base in exponential functions that model contextual scenarios involving continuously compounded interest.
3.1.2 Express an exponential function in an equivalent form to reveal properties of the graph and/or the contextual scenario.	**3.1.2a** Any exponential model can be expressed in any base, including the natural base, using properties of exponents and/or function composition and properties of logarithms. **3.1.2b** A horizontal translation of the graph of an exponential function can also be thought of as a vertical dilation of the graph because $f(x)=b^{(x+k)}$ can be expressed as $f(x)=b^x \bullet b^k$, where b^k is a constant. **3.1.2c** A horizontal dilation of the graph of an exponential function is equivalent to a change of the base of the function, because $f(x)=b^{kx}$ can be expressed as $f(x)=\left(b^k\right)^x$, where b^k is a constant.
3.1.3 Construct a representation of a logarithmic function.	**3.1.3a** The graph of a logarithmic function exhibits vertical asymptotic behavior. **3.1.3b** An algebraic representation of a logarithmic function with base *b* is a transformation of $f(x)=\log_b(x)$, where $b \neq 1$ and $b>0$. **3.1.3c** A verbal representation of a logarithmic function describes additive changes in the output corresponding to multiplicative changes in the input. **3.1.3d** Data sets and/or contextual scenarios that exhibit a roughly constant ratio of the inputs for equal additive changes in the outputs are appropriately modeled by logarithmic functions.
3.1.4 Express a logarithmic function in an equivalent form to reveal properties of the graph and/or the contextual scenario.	**3.1.4a** A logarithmic function can be expressed in any base using properties of logarithms. Logarithmic functions are often expressed in base *e*, which is called the natural logarithm and is notated as "ln" rather than $\log_e$. **3.1.4b** A horizontal dilation is equivalent to a vertical translation because $f(x)=\log_b(kx)$ can be rewritten as $f(x)=\log_b(k)+\log_b(x)$, where $\log_b(k)$ is a constant. **3.1.4c** Raising the input of a logarithmic function to a power of *k* results in a vertical dilation of the graph because $f(x)=\log_b\left(x^k\right)$ can be rewritten as $f(x)=k \bullet \log_b(x)$.
3.1.5 Construct a representation of the inverse function of an exponential or logarithmic function.	**3.1.5a** The inverse of the exponential function $f(x)=b^x$ is the logarithmic function $g(x)=\log_b(x)$. **3.1.5b** The inverse of the logarithmic function $f(x)=\log_b(x)$ is the exponential function $g(x)=b^x$.

Learning Objectives *Students will be able to …*	**Essential Knowledge** *Students need to know that …*
3.1.6 Solve equations involving exponential or logarithmic functions, including those arising from contextual scenarios.	**3.1.6a** Equations involving exponential functions can be solved algebraically by taking a logarithm or have solutions that can be estimated by examining a graph of the function. **3.1.6b** Equations involving logarithmic functions can be solved algebraically by exponentiating or have solutions that can be estimated by examining a graph of the function.

Content Boundary: Because students have familiarity with exponential functions from Pre-AP Algebra 1 and Unit 1 of this course, the exponential functions in this key concept should primarily include the natural base, *e*.

Cross Connection: This key concept provides an opportunity for students to use what they learned about exponential functions in Pre-AP Algebra 1 and function transformations, composition, and inverses in Unit 2 to deepen their understanding of exponential functions, a critically important function family.

Cross Connection: Logarithms are an essential tool in advanced mathematics courses. They are used in AP Statistics to transform a data set into a set that displays a more linear trend than the original one, which makes the data set easier to model. In AP Calculus, logarithms are often used in solving differential equations, especially those involving population growth, which is often exponential.

KEY CONCEPT 3.2: POLYNOMIAL AND RATIONAL FUNCTIONS

Using functions to make sense of sums and quotients of powers

Learning Objectives *Students will be able to …*	**Essential Knowledge** *Students need to know that …*
3.2.1 Construct a representation of a polynomial function.	**3.2.1a** A polynomial function is a function whose algebraic representation can be expressed as $f(x)=a_n x^n + a_{n-1}x^{n-1}+\cdots+a_1 x+a_0$, where $a_n \neq 0$. **3.2.1b** Scenarios involving areas of figures and surface areas and volumes of solids are often well modeled by polynomial functions. **3.2.1c** Data sets and/or contextual scenarios that exhibit maximum or minimum values are sometimes well modeled by polynomial functions. **3.2.1d** Data sets and/or contextual scenarios where equal changes in the input values correspond to approximately constant nth differences in the output values are often well modeled by polynomial functions of degree n.
3.2.2 Express a polynomial function in an equivalent algebraic form to reveal properties of the function.	**3.2.2a** The standard form of a polynomial function reveals the degree of the polynomial, n, which is the highest power of all the terms. **3.2.2b** A linear factor, $(x-a)$, of a polynomial function, p, corresponds to a zero (or root) of p at $x=a$ because $p(a)=0$. **3.2.2c** A polynomial function factored into a product of linear factors reveals the x-intercepts of the graph of the function, which are the real zeros of the polynomial. The total number of real zeros is at most equal to the degree of the polynomial function.
3.2.3 Identify key features of the graph of a polynomial function.	**3.2.3a** A local maximum or minimum of a nonconstant polynomial function corresponds to the output value of the point at which the function switches between increasing to decreasing in either order. **3.2.3b** Between every two real zeros of a nonconstant polynomial function, there must be at least one input value corresponding to a local maximum or minimum. **3.2.3c** The end behavior of a polynomial function can be determined visually from its graph or by examining the degree of the polynomial and the sign of its leading coefficient. **3.2.3d** If a linear factor $(x-a)$ of a polynomial function has an even power, then the signs of the output values are the same for input values near $x=a$. For these polynomials, the graph will be tangent to the x-axis at $x=a$.
3.2.4 Perform arithmetic with complex numbers.	**3.2.4a** Every complex number has the form $a+bi$ where a and b are real numbers and $i^2=-1$. The real part of the complex number is a and the imaginary part of the complex number is b. **3.2.4b** The real numbers are a subset of the complex numbers since every real number a is equivalent to the complex number $a+0i$.

Learning Objectives *Students will be able to …*	**Essential Knowledge** *Students need to know that …*
3.2.4 Perform arithmetic with complex numbers. *(continued)*	**3.2.4c** Adding or subtracting two complex numbers involves performing the indicated operation with the real parts and the imaginary parts separately. Multiplying two complex numbers is accomplished by applying the distributive property and using the relationship $i^2 = -1$. **3.2.4d** Complex numbers occur naturally as solutions to quadratic equations with real coefficients. Therefore, verifying that a complex number is a solution of a quadratic equation requires adding and multiplying complex numbers.
3.2.5 Construct a representation of a rational function.	**3.2.5a** A rational function is a function whose algebraic form consists of the quotient of two polynomial functions. **3.2.5b** Quantities that are inversely proportional are often well modeled by rational functions. For example, both the magnitudes of gravitational force and electromagnetic force between objects are inversely proportional to their squared distance.
3.2.6 Identify key features of the graph of a rational function.	**3.2.6a** Rational functions often have restricted domains. These restrictions correspond to the zeros of the polynomial in the denominator and often manifest in the graph as vertical asymptotes. **3.2.6b** Zeros of a rational function correspond to the zeros of the polynomial in the numerator that are in the domain of the function. The x-intercepts of the graph of a rational function correspond to the zeros of the function. **3.2.6c** The end behavior of a rational function can be determined by examining the behavior of a function formed by the ratio of the leading term of the numerator to the leading term of the denominator.

Content Boundary: For complex number arithmetic, students are expected to add, subtract, and multiply two complex numbers including squaring a complex number. Rationalizing the denominator of expressions that involve complex numbers—that is, dividing complex numbers—is beyond the scope of this course.

Content Boundary: Factoring polynomial expressions of degree greater than 2 is beyond the scope of the course. For polynomials of degree 3 or greater, factorizations or graphs should be provided if students are expected to find the zeros of the polynomial.

Content Boundary: Analyzing rational functions that have a common factor in the numerator and denominator—that is, rational functions with "holes"—is beyond the scope of the course.

Cross Connection: The content of this key concept connects back to students' experience with linear and quadratic functions from Pre-AP Algebra 1, because linear functions are polynomial functions of degree 1 and quadratic functions are polynomial functions of degree 2.

Cross Connection: Polynomial functions have been used throughout history as imperfect models of contextual scenarios that have maximum and minimum values. Taking the derivative of a polynomial function is straightforward which makes them attractive models for scenarios that involve rates of change.

Cross Connection: In AP Calculus, rational functions are analyzed through the concept of a limit because their graphs often have interesting asymptotic properties.

KEY CONCEPT 3.3: SQUARE ROOT AND CUBE ROOT FUNCTIONS

Using functions to make sense of inverting quadratic and cubic relationships

Learning Objectives *Students will be able to ...*	**Essential Knowledge** *Students need to know that ...*
3.3.1 Construct a representation of a square root function.	**3.3.1a** The square root function, $f(x)=\sqrt{x}$, is the inverse of the quadratic function $g(x)=x^2$ over the restricted domain $[0,\infty)$. Therefore, the graph of $y=\sqrt{x}$ resembles the graph of $y=x^2$ for $x\geq 0$ reflected across the line $y=x$. **3.3.1b** The domain of a square root function, a function transformation of $f(x)=\sqrt{x}$, corresponds to the set of input values for which the expression under the radical is nonnegative. **3.3.1c** Real-world scenarios involving distance traveled and elapsed time for free-falling objects are well modeled by square root functions.
3.3.2 Construct a representation of a cube root function.	**3.3.2a** The cube root function, $f(x)=\sqrt[3]{x}$, is the inverse of the cubic function $g(x)=x^3$. Thus, the graph of $y=\sqrt[3]{x}$ resembles the graph of $y=x^3$ reflected across the line $y=x$. **3.3.2b** The domain of a cube root function, a function transformation of $f(x)=\sqrt[3]{x}$, is all real numbers. **3.3.2c** Real-world scenarios involving side lengths of solids with a known volume are well modeled by cube root functions.
3.3.3 Construct a representation of the inverse function of a given quadratic function.	**3.3.3a** The inverse of a quadratic function is a square root function. **3.3.3b** The algebraic representation of the inverse of a quadratic function $y=f(x)$ is determined by first expressing f in vertex form and then using inverse operations to express x in terms of y. **3.3.3c** Since many output values of quadratic functions are each associated with multiple input values, constructing an inverse of a quadratic function requires restricting the domain of the function so the function is invertible. For values of x in the constrained domain of the quadratic function f, $f^{-1}(f(x))=x$.
3.3.4 Solve equations involving square root or cube root functions, including those arising from contextual scenarios.	**3.3.4a** Equations involving square roots and cube roots arising from contextual scenarios can be solved algebraically using inverse operations, such as squaring or cubing, or have solutions that can be estimated by examining an associated graph of the function models. **3.3.4b** Solving equations by squaring can introduce values called extraneous solutions, which are not actual solutions of the equation.

Cross Connection: The square root function appears in a variety of applications, most notably in scenarios involving distance or velocity as a function of time. These types of problems could involve finding the distance between two points in the plane using the Pythagorean theorem or finding a typical distance between a data value and the mean of the data set, called the standard deviation of a data set. In both cases, taking the square root is used to undo squaring a quantity.

Unit 4T: Trigonometric Functions

Suggested timing: Approximately 6 weeks

This unit provides an exploration of trigonometric functions. Trigonometry is the branch of mathematics that connects two fundamental geometric objects: triangles and circles. In Pre-AP Geometry with Statistics, students learned that the trigonometric ratios relate acute angle measures to ratios of side lengths in right triangles. Algebra 2 extends those relationships to include all real numbers. When the domains of trigonometric functions include angle measures greater than 90°, including greater than 360°, these functions are far more useful in modeling contextual scenarios that involve periodic phenomena, such as the rotation of a ceiling fan, the height of a Ferris wheel car, or the ebb and flow of tides. A sound understanding of trigonometric functions demystifies these common real-world contexts, putting the power of mathematical reasoning at students' command.

Beginning with the introduction of radians as units of angle measure, the unit continues with an investigation of the sine and cosine functions and their transformations, collectively referred to as sinusoidal functions. Students then use what they know about sinusoidal functions and properties of quotients of functions to understand the properties of the tangent function and the three reciprocal trigonometric functions. Finally, students use inverse trigonometric functions to solve problems related to circular and periodic motion. Please note that the Pre-AP three-year mathematics sequence includes trigonometry in Algebra 2 to create a more equitable pathway for students who take Algebra 1 in 9th grade to potentially enroll in AP Calculus AB in 12th grade. If a state's standards do not require trigonometry in Algebra 2, then the school may use **Unit 4M: Matrices and Their Applications** to fulfill their Pre-AP commitments.

ENDURING UNDERSTANDINGS

Students will understand that …

- Trigonometry connects the study of circles and the study of right triangles.
- Real-world contexts that exhibit periodic behavior or circular motion can be modeled by trigonometric functions.

KEY CONCEPTS

- 4T.1: Radian Measure and Sinusoidal Functions – Using circles and triangles to make sense of periodic phenomena
- 4T.2: The Tangent Function and Other Trigonometric Functions – Using quotients of trigonometric functions to define new functions
- 4T.3: Inverting Trigonometric Functions – Using trigonometry to solve equations involving circular motion and periodic phenomena

KEY CONCEPT 4T.1: RADIAN MEASURE AND SINUSOIDAL FUNCTIONS

Using circles and triangles to make sense of periodic phenomena

Learning Objectives *Students will be able to ...*	**Essential Knowledge** *Students need to know that ...*
4T.1.1 Use the radian measure of an angle to relate the radius of a circle to the length of the arc subtended by that angle.	**4T.1.1a** The radian measure of an angle expresses the ratio of the subtended arc length to the radius of the circle in which it is a central angle. **4T.1.1b** An angle that has a measure of 1 radian cuts off an arc length equal to the length of the radius. **4T.1.1c** There is a proportional relationship between the radian measure of an angle, the subtended arc length, and the radius length. For a given angle, the ratio of the subtended arc length to the radius length is constant for any radius of the circle in which the angle is a central angle.
4T.1.2 Determine when two angles in the coordinate plane are coterminal.	**4T.1.2a** Angle measures can be expressed in radians or in degrees. The relationship between these units is proportional such that $\frac{\text{measure in degrees}}{\text{measure in radians}} = \frac{360}{2\pi}$. **4T.1.2b** In the coordinate plane, an angle is in standard position when its vertex is at the origin and one of its rays lies along the positive x-axis. Its other ray is called the terminal ray. **4T.1.2c** Two angles in standard position are coterminal if their terminal rays coincide. The amount of rotation of coterminal angles may differ by an integer number of revolutions. **4T.1.2d** Positive angle measures indicate that the terminal ray of the angle is constructed by the counterclockwise rotation of the ray about the origin. Negative angle measures indicate the terminal ray of the angle is constructed by a clockwise rotation of the ray about the origin.
4T.1.3 Construct a representation of a sinusoidal function.	**4T.1.3a** Periodic phenomena have repeating patterns of output values. Aspects of these phenomena can often be appropriately modeled by sinusoidal functions. **4T.1.3b** A unit circle has a radius of 1 unit of measure. The unit of measure should be determined by the context. **4T.1.3c** In the context of circular motion, the function $f(\theta) = \sin(\theta)$ relates the measure of an angle in standard position to the vertical displacement from the origin of a point on the unit circle and the function $f(\theta) = \cos(\theta)$ relates the measure of an angle in standard position to the horizontal displacement from the origin of a point on the unit circle. **4T.1.3d** Sinusoidal functions include the functions $f(x) = \sin(x)$ and $f(x) = \cos(x)$ and their transformations.

Learning Objectives *Students will be able to ...*	**Essential Knowledge** *Students need to know that ...*
4T.1.4 Determine the exact coordinates of any point on a circle centered at the origin.	**4T.1.4a** The coordinates of the point at which the terminal ray of an angle in standard position intersects a unit circle with radius r are uniquely determined by the measure of that angle, θ, where $(x, y) = (r\cos(\theta), r\sin(\theta))$. For points on a unit circle, the coordinates are given by $(x, y) = (\cos(\theta), \sin(\theta))$. **4T.1.4b** The reference triangle for a given point on a circle is a right triangle whose three vertices are the origin, the point itself, and a point on the x-axis. The reference triangle can be useful in determining the exact coordinates of the given point.
4T.1.5 Identify key characteristics of a sinusoidal function.	**4T.1.5a** The amplitude of a sinusoidal function is half the difference between its maximum and minimum values. **4T.1.5b** The period of a sinusoidal function is the length of the interval of the input values over which it completes one rotation. The frequency of a sinusoidal function is the number of periods within an interval of length 2π. **4T.1.5c** Additive transformations of sinusoidal functions result in vertical or horizontal translations of the graph. A horizontal translation of a sinusoidal function is called a phase shift. **4T.1.5d** Multiplicative transformations of sinusoidal functions result in vertical or horizontal dilations of the graph. These transformations can impact the amplitude, period, or frequency of the sinusoidal function.
4T.1.6 Construct a sinusoidal function to model a periodic phenomenon that has a specified frequency, period, amplitude, and phase shift.	**4T.1.6a** The smallest interval of the input values over which the maximum or minimum output values start to repeat can be used to determine or estimate the period and frequency of the sinusoidal function model. **4T.1.6b** The maximum and minimum output values can be used to determine or estimate the amplitude for a sinusoidal function model. **4T.1.6c** The reference point for the input quantity can be used to determine or estimate a phase shift for the sinusoidal function model.
4T.1.7 Solve problems involving trigonometric identities.	**4T.1.7a** The Pythagorean theorem for trigonometric functions, $\sin^2(\theta) + \cos^2(\theta) = 1$, can be deduced from the fact that a circle of radius r centered at the origin is the solution set to the equation $x^2 + y^2 = r^2$ and that the coordinates of a point on that circle are given by $x = r\cos(\theta)$ and $y = r\sin(\theta)$.

Content Boundary: The double-angle and half-angle formulas are beyond the scope of the course because of their limited usefulness in developing an understanding of trigonometric functions. The reciprocal identities are beyond the scope of this key concept but are introduced in Key Concept 4T.2. Students are expected to use the Pythagorean identity to solve problems involving squared trigonometric expressions, but verifying trigonometric identities is beyond the scope of the course.

Cross Connection: Embedding an angle in the coordinate plane allows students to connect their Pre-AP Geometry with Statistics experience with the concept of an angle, a union of two rays, to the concept of rotation. The measure of an angle in the coordinate plane quantifies the amount of the rotation about the vertex required of the first ray to coincide with the second ray, while the sign of the angle measure indicates the direction of this rotation.

KEY CONCEPT 4T.2: THE TANGENT FUNCTION AND OTHER TRIGONOMETRIC FUNCTIONS

Using quotients of trigonometric functions to define new functions

Learning Objectives *Students will be able to …*	**Essential Knowledge** *Students need to know that …*
4T.2.1 Construct a representation of a tangent function.	**4T.2.1a** For an angle in standard position, the tangent of the angle's measure is the slope of the terminal ray. **4T.2.1b** When the terminal ray is vertical, the slope of the ray is undefined. Therefore, the domain of the tangent function is restricted to exclude all values that correspond to vertical terminal rays. In other words, the domain of $f(\theta)=\tan(\theta)$ excludes all values of θ such that $\theta=\frac{\pi}{2}+k\pi$ for integer values of k. **4T.2.1c** When two angles in standard position have measures that differ by π radians, their terminal rays form a line and the rays have the same slope. As such, the tangent function has a period of π. **4T.2.1d** Contextual scenarios involving the height of a rising or falling object or slopes of lines are often appropriately modeled with a tangent function.
4T.2.2 Identify key characteristics and values of functions that are defined by quotients of sinusoidal functions.	**4T.2.2a** The tangent of angle θ is the slope of the terminal ray, which passes through the points (0, 0) and $(r\cos(\theta), r\sin(\theta))$. Thus, $\tan(\theta)=\frac{\sin(\theta)}{\cos(\theta)}$. **4T.2.2b** Secant, cosecant, and cotangent are the names given to trigonometric functions formed by quotients of sinusoidal functions, defined as $\sec(\theta)=\frac{1}{\cos(\theta)}$, $\csc(\theta)=\frac{1}{\sin(\theta)}$, and $\cot(\theta)=\frac{\cos(\theta)}{\sin(\theta)}$.

Content Boundary: Students develop an understanding of the properties of tangent, secant, cosecant, and cotangent functions and their relationships to the sinusoidal functions through this key concept. Verifying trigonometric identities is beyond the scope of the course.

Cross Connection: This key concept connects the slope of a line to the tangent of an angle that the line forms with the horizontal axis. This connection provides an opportunity for students to build a deeper understanding of linear relationships and link trigonometry to the content of Pre-AP Algebra 1. A comprehensive understanding of slope supports students in accessing the concept of the derivative of a function in AP Calculus.

KEY CONCEPT 4T.3: INVERTING TRIGONOMETRIC FUNCTIONS

Using trigonometry to solve equations involving circular motion and periodic phenomena

Learning Objectives *Students will be able to ...*	**Essential Knowledge** *Students need to know that ...*
4T.3.1 Construct a representation of an inverse trigonometric function.	**4T.3.1a** The inputs and outputs of inverse trigonometric functions are switched from their corresponding trigonometric functions, so the output of an inverse trigonometric function is often interpreted as an angle measure and the input is a value in the range of the corresponding trigonometric function. **4T.3.1b** The inverse trigonometric functions arcsine, arccosine, and arctangent (also represented as $\sin^{-1}$, $\cos^{-1}$, and $\tan^{-1}$) are defined by restricting the domain of the sine function to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, restricting the domain of the cosine function to $[0, \pi]$, and restricting the domain of the tangent function to $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ so the trigonometric functions are invertible on their restricted domains.
4T.3.2 Solve equations involving trigonometric functions.	**4T.3.2a** For algebraic equations involving trigonometric expressions, inverse trigonometric functions can be used to determine the inputs corresponding to a specified output. **4T.3.2b** There could be multiple solutions to a trigonometric equation. The exact number of solutions is determined by the context.

Content Boundary: Solving a trigonometric equation for all possible solutions is beyond the scope of this course. Problems and exercises involving trigonometric equations used to model periodic phenomena should be limited to solutions over a finite interval.

Cross Connection: The names of inverse trigonometric functions include the prefix *arc-* because the output of each of these functions is an arc length measured in radians. Knowing why the prefix *arc-* is used can help students connect inverse trigonometric functions to circles, as explored in Key Concept 4T.1.

Unit 4M: Matrices and Their Applications

Suggested timing: Approximately 6 weeks

The final unit in this course provides an exploration of matrices and their applications. Matrices are useful tools for organizing information and are used in career fields as diverse as engineering, business, and statistics. The goal of the unit is for students to understand how to construct and use matrices to apply certain geometric transformations and solve systems of linear equations. It is important to note that the operations performed on matrices in this course are limited to multiplication and finding the nth power of a square matrix. Addition and subtraction of matrices are not included in this unit because these operations are not necessary for applying geometric transformations or solving systems of linear equations using matrices. Also, most of the operations performed on matrices in this unit are expected to be performed with technology.

In this unit, students build upon their previous experience with functions, inverse functions, and transformations in the plane to support their understanding of how certain geometric transformations can be represented with matrix multiplication. Students use the inverse of a transformation matrix to identify the preimage of a specified point under a given linear transformation. Students use the inverse of a matrix to solve a system of linear equations.

The unit culminates in an exploration of recursive processes that model the relationship between the sizes of subgroups of a partitioned population. Using real-world scenarios, students apply matrix operations to determine the previous and future sizes of these subgroups and observe that their sizes stabilize over time.

ENDURING UNDERSTANDINGS

Students will understand that …

- A matrix is a tool that can be used to efficiently organize mathematical information.
- Matrix multiplication provides efficient ways to evaluate multiple linear expressions simultaneously.
- Systems of linear equations can be solved using inverse matrices.

KEY CONCEPTS

- 4M.1: Geometric Transformations – Using matrices to determine images and preimages of points in the coordinate plane
- 4M.2: Solving Systems of Equations with Matrices – Solving systems of equations with inverse matrices
- 4M.3: Applications of Matrix Multiplication – Modeling how subgroups of a population move in a predictable way

KEY CONCEPT 4M.1: GEOMETRIC TRANSFORMATIONS

Using matrices to determine images and preimages of points in the coordinate plane

Learning Objectives *Students will be able to …*	**Essential Knowledge** *Students need to know that …*
4M.1.1 Identify whether a geometric transformation is a linear transformation.	**4M.1.1a** Given a geometric transformation, T, the notation $T(x, y)$ represents the image of point (x, y). **4M.1.1b** A geometric transformation that can be expressed in the form $T(x, y) = (ax + by, cx + dy)$ is called a linear transformation. **4M.1.1c** A linear transformation is a function that maps a point (x, y) to its image. **4M.1.1d** In any linear transformation, the origin maps to the origin, and lines map to lines.
4M.1.2 Express a system of linear expressions using multiple representations.	**4M.1.2a** The sum of terms consisting of the product of a constant and an unknown raised to the first power is called a linear expression (e.g., $ax + by$ or $ax + by + cz$). **4M.1.2b** The coefficients of a system of linear expressions can be organized as entries in a matrix. The entries in each column describe constraints on one of the variables and the entries in each row correspond to the coefficients of each expression. **4M.1.2c** An algebraic representation of a linear transformation is a system of linear expressions that defines the coordinates of the preimage. This is expressed as $T(x, y) = (ax + by, cx + dy)$. **4M.1.2d** The linear transformation $T(x, y) = (ax + by, cx + dy)$ can be expressed using matrix multiplication as $T(x, y) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$.
4M.1.3 Multiply two matrices when the product is defined, both with and without technology.	**4M.1.3a** A matrix with m rows and n columns is said to have dimension $m \times n$. If the number of rows is equivalent to the number of columns, then the matrix is called a square matrix. **4M.1.3b** Given two matrices, A and B, the product AB is defined if the number of columns in matrix A is equal to the number of rows in matrix B. The resulting matrix, AB, has the same number of rows as matrix A and the same number of columns as matrix B. **4M.1.3c** Given the product of two matrices, where $AB = C$, the entry in the ith row and jth column of matrix C is determined by the sum of the products of the corresponding entries in the ith row of matrix A and the jth column of matrix B. **4M.1.3d** Given two matrices, A and B, the product AB is not necessarily equal to BA. That is, matrix multiplication is not commutative.

Learning Objectives *Students will be able to …*	**Essential Knowledge** *Students need to know that …*
4M.1.4 Determine the image of a set of points under a linear transformation or a sequence of linear transformations.	**4M.1.4a** For a linear transformation defined as $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the product $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ is equivalent to $\begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}$. The product can be interpreted as the coordinate pair $(ax+by, cx+dy)$, which is the image of the point (x, y). **4M.1.4b** The set of n points $(x_1, y_1)\ldots(x_n, y_n)$ can be represented as the $2 \times n$ matrix $X = \begin{bmatrix} x_1 & \cdots & x_n \\ y_1 & \cdots & y_n \end{bmatrix}$. The image of these n points is given by AX where A is a 2×2 matrix that represents a linear transformation. **4M.1.4c** If matrices A and B represent two linear transformations and X is the $2 \times n$ coordinate matrix, then these transformations can be composed by first applying linear transformation A, which is given by AX, and then linear transformation B. The image of this sequence of transformations is given by $B \bullet (AX)$. This composition is described by the product $B \bullet (AX)$ since matrix multiplication is not commutative. **4M.1.4d** A square matrix with entries of 1 along the main diagonal and all other entries of 0 is called the identity matrix. The 2×2 identity matrix defines the identity transformation and maps all points in the plane onto themselves.
4M.1.5 Interpret the determinant of a 2×2 matrix geometrically.	**4M.1.5a** The determinant of a 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is given by $ad - bc$. The determinant of matrix A can be represented as $\det(A)$. **4M.1.5b** Under a linear transformation represented by $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the image of the square with vertices at (0, 0), (0, 1), (1, 0), and (1, 1) is a parallelogram with vertices at (0, 0), (a, c), (b, d), and $(a+b, c+d)$. This parallelogram has an area of $\lvert ad - bc \rvert$ square units, which is the absolute value of the determinant of the transformation matrix. **4M.1.5c** A linear transformation will map a polygon in the plane to another polygon in the plane. The area of the image can be determined by scaling the area of the preimage by the absolute value of the determinant of the matrix that represents the linear transformation.
4M.1.6 Express a sequence of linear transformations as a single linear transformation.	**4M.1.6a** Matrix multiplication is associative; that is, $A \bullet (BX) = (AB) \bullet X$. **4M.1.6b** If two linear transformations are given by matrices A and B, then the linear transformation AB is equivalent to first applying linear transformation B and then applying linear transformation A.

Learning Objectives *Students will be able to …*	**Essential Knowledge** *Students need to know that …*
4M.1.7 Determine the inverse of a square matrix, including matrices describing linear transformations, both with and without technology.	**4M.1.7a** Given a square matrix, A, its inverse, if it exists, is another square matrix, A^{-1}. The products AA^{-1} and $A^{-1}A$ are both equal to the identity matrix. **4M.1.7b** For a 2×2 matrix A, if det(A) is nonzero then the inverse matrix A^{-1} exists. If det(A) is nonzero then the inverse of A is given by $\frac{1}{\det(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.
4M.1.8 Determine the preimage of a specified point under a given linear transformation.	**4M.1.8a** If T and T^{-1} are inverse linear transformations, then $T(x, y) = (a, b)$ implies $T^{-1}(a, b) = (x, y)$. The matrix that describes linear transformation T^{-1} is the inverse of the matrix that describes the linear transformation T. **4M.1.8b** Matrix multiplication on the left by A^{-1} is the inverse operation of matrix multiplication on the left by A. **4M.1.8c** If matrix A represents a linear transformation and maps point (x_1, y_1) to (x_2, y_2), then solving the equation $A\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$ for $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ will give the preimage of the point (x_2, y_2). This equation can be solved by multiplying each side of the equation on the left by A^{-1}, assuming A^{-1} exists.

Content Boundary: Geometric transformations are limited to two-dimensional transformations. Students focus on characteristics of linear transformations and how these transformations act on the entire plane.

Content Boundary: Students should be able to perform matrix multiplication by hand for matrices with dimensions $(n \times n)(n \times 1)$ and $(2 \times 2)(2 \times 2)$ and use technology to multiply matrices of other dimensions.

Cross Connection: This key concept builds directly on students' understanding from Pre-AP Geometry with Statistics of performing transformations on points in the plane and expressing transformations using function notation.

Cross Connection: In AP Precalculus, students revisit topics from this key concept as they make sense of matrices acting as functions on vectors. In this course, the focus is on students understanding how matrices act as functions on points.

Cross Connection: In AP Computer Science A, students apply basic understandings of this key concept when they represent matrices as two-dimensional arrays in a programming language.

KEY CONCEPT 4M.2: SOLVING SYSTEMS OF EQUATIONS WITH MATRICES

Solving systems of equations with inverse matrices

Learning Objectives *Students will be able to …*	**Essential Knowledge** *Students need to know that …*
4M.2.1 Express a system of linear equations using multiple representations.	**4M.2.1a** A linear equation is a linear expression that is equal to a constraint (e.g., $ax + by + cz = d$). Linear equations can have more than two unknowns. **4M.2.1b** A verbal representation of a system of linear equations describes the relationship between the unknowns and the constraints on how the unknown values can vary. Verbal representations often describe contextual scenarios. **4M.2.1c** The system of linear equations given by $ax + by = e$ and $cx + dy = f$ can be expressed in an equivalent form as the matrix equation $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$.
4M.2.2 Solve a system of linear equations using inverse matrices.	**4M.2.2a** The solution to the system of linear equations represented by $AX = B$ is $X = A^{-1}B$, assuming A^{-1} exists. **4M.2.2b** The matrix equation $AX = B$ has a unique solution when det(A) is nonzero because the inverse, A^{-1}, is defined. If det(A) = 0, then A^{-1} is not defined and the matrix equation $AX = B$ has either no solution or infinitely many solutions. **4M.2.2c** The solution to a system of linear equations expressed with matrices derived from a context must be interpreted through that context for its meaning to be understood.
4M.2.3 Construct the algebraic representation of a polynomial function that passes through a set of given points using technology.	**4M.2.3a** Given n points, a polynomial of at most degree $n - 1$ that passes through those points can be found by solving for coefficients of that polynomial. **4M.2.3b** Substituting the coordinates of n points into a polynomial of degree $n - 1$ of the form $y = a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1x + a_0$ produces a system of n linear equations where the coefficients are the unknowns. **4M.2.3c** A system of n linear equations, where the coefficients are the unknowns, can be solved by writing the system of n linear equations as a matrix equation and then using technology to solve that matrix equation.

Content Boundary: Students are expected to calculate the determinant of a 2×2 matrix by hand. The process for evaluating the determinant of a higher-order square matrix by hand is out of scope and should be done using technology.

Cross Connection: This key concept builds on students' experiences with solving systems of equations and modeling with systems of equations from Pre-AP Algebra 1. Through this unit, students extend their problem-solving options by using matrices to solve systems of equations.

Cross Connection: In AP Precalculus, students are expected to use matrices to solve systems of linear equations to minimize the use of algebraic techniques commonly used in Pre-AP Algebra 1.

KEY CONCEPT 4M.3: APPLICATIONS OF MATRIX MULTIPLICATION

Modeling how subgroups of a population move in a predictable way

Learning Objectives *Students will be able to …*	**Essential Knowledge** *Students need to know that …*
4M.3.1 Express a recursive process that models the relationship between two or three related subgroups of a population using multiple representations.	**4M.3.1a** Populations that are partitioned into subgroups whose members move among these subgroups in predictable ways are well modeled by matrices and transition diagrams. **4M.3.1b** A transition diagram represents the subgroups of a population and uses arrows to describe how the members of the population move between these subgroups in predictable ways. **4M.3.1c** In the equation $AX_0 = X_1$, the entries of matrix A describe how members of a population move between subgroups over 1 unit of time, matrix X_0 describes the size of each subgroup at a specified moment, and matrix X_1 describes the size of each subgroup 1 time-unit away. **4M.3.1d** Matrix multiplication can be used to determine the sizes of subgroups of a population at specific moments of time.
4M.3.2 Determine the future sizes of subgroups of a population given the current size of each subgroup.	**4M.3.2a** The entries of the matrix A^n quantify how members of the population move between its subgroups over an n-unit period of time. **4M.3.2b** The matrix A^n is determined by calculating the product of n copies of matrix A. The matrix A^n does not result from raising each entry of matrix A to the nth power. **4M.3.2c** Given an $m \times 1$ matrix containing the size of each subgroup of the population at time k, X_k, the matrix expression $A^n X_k$ gives the size of each subgroup n time-units from time k. This matrix expression is equivalent to the $m \times 1$ matrix X_{k+n}.
4M.3.3 Identify how the sizes of related subgroups of a population stabilize over time.	**4M.3.3a** For a given population partitioned into two or three subgroups, each subgroup will approach a constant size over time even as members of the population continue to move between subgroups. **4M.3.3b** For a population partitioned into two subgroups, the ratio of the sizes of pairs of subgroups at a particular moment will approach a constant value over time. This ratio is independent of size of the population.
4M.3.4 Determine the previous sizes of subgroups of a population given the current size of each subgroup.	**4M.3.4a** Solving the equation $AX_0 = X_1$ for X_0 will give the previous size of each subgroup of a population.

Content Boundary: A primary focus for this key concept is to use random processes to generate values so that the entries in the coefficient matrix of the matrix equation are probabilities.

Content Boundary: Although not all systems with three subpopulations stabilize, the systems selected as examples in for this unit will stabilize.

Cross Connection: Matrices are used as transformations in both this course and AP Precalculus. In this course, these matrices act on points in the plane, while in AP Precalculus, they act on vectors.

Pre-AP Algebra 2 Model Lessons

Model lessons in Pre-AP Algebra 2 are developed in collaboration with Algebra 2 educators across the country and are rooted in the course framework, shared principles, and areas of focus. Model lessons are carefully designed to illustrate on-grade-level instruction. Pre-AP strongly encourages teachers to internalize the lessons and then offer the supports, extensions, and adaptations necessary to help all students achieve the lesson goals.

The purpose of these model lessons is twofold:

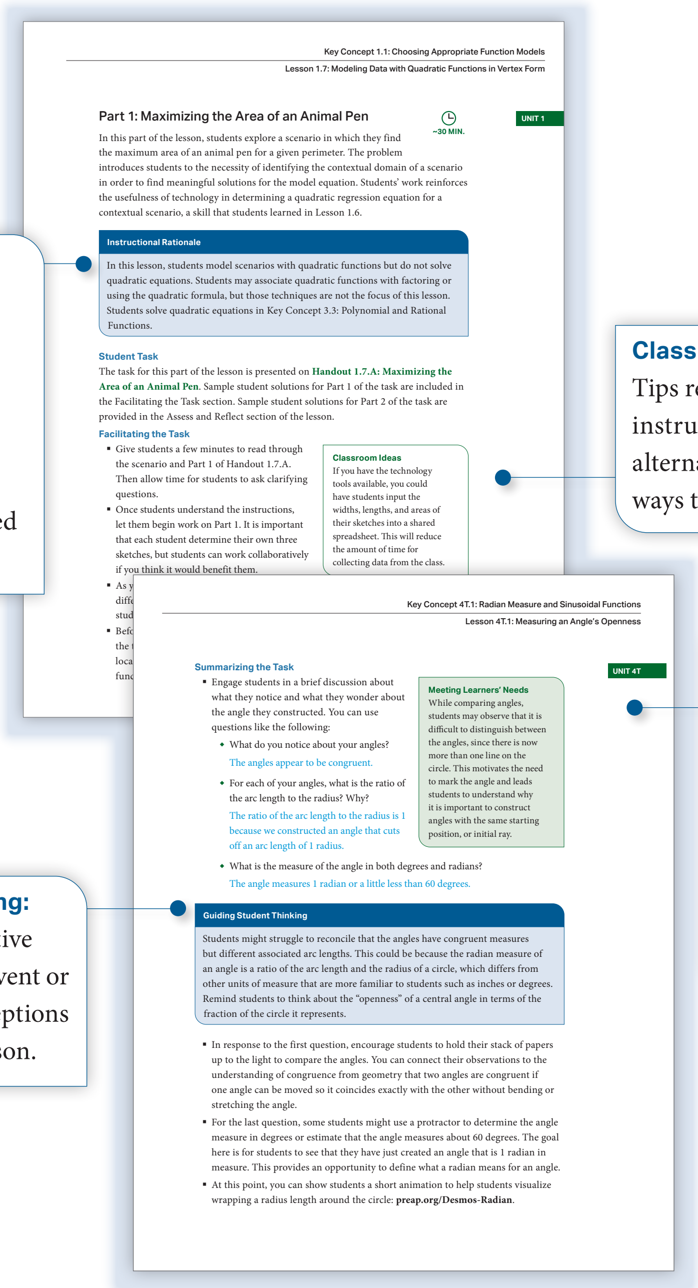
- **Robust instructional support for teachers:** Pre-AP Algebra 2 model lessons are comprehensive lesson plans that, along with accompanying student resources, embody the Pre-AP approach to teaching and learning. Model lessons provide clear and substantial instructional guidance to support teachers as they engage students in the shared principles and areas of focus.
- **Key instructional strategies:** Commentary and analysis embedded in each lesson highlight not just what students and teachers do in the lesson, but also how and why they do it. This educative approach provides a way for teachers to gain unique insight into key instructional moves that are powerfully aligned with the Pre-AP approach to teaching and learning. In this way, each model lesson works to support teachers in the moment of use with students in their classroom.

Teachers have the option to use any or all model lessons alongside their own locally developed instructional resources. Model lessons target content areas that tend to be challenging for teachers and students. While the lessons are distributed throughout all four units, they are concentrated more heavily in the beginning of the course to support teachers and students in establishing a strong foundation in the Pre-AP approach to teaching and learning.

SUPPORT FEATURES IN MODEL LESSONS

The following support features recur throughout the Pre-AP Algebra 2 lessons to promote teacher understanding of the lesson design and provide direct-to-teacher strategies for adapting lessons to meet their students' needs:

- Instructional Rationale
- Guiding Student Thinking
- Classroom Ideas
- Meeting Learners' Needs



Key Concept 1.1: Choosing Appropriate Function Models
Lesson 1.7: Modeling Data with Quadratic Functions in Vertex Form

UNIT 1

Part 1: Maximizing the Area of an Animal Pen

~30 MIN.

In this part of the lesson, students explore a scenario in which they find the maximum area of an animal pen for a given perimeter. The problem introduces students to the necessity of identifying the contextual domain of a scenario in order to find meaningful solutions for the model equation. Students' work reinforces the usefulness of technology in determining a quadratic regression equation for a contextual scenario, a skill that students learned in Lesson 1.6.

Instructional Rationale

In this lesson, students model scenarios with quadratic functions but do not solve quadratic equations. Students may associate quadratic functions with factoring or using the quadratic formula, but those techniques are not the focus of this lesson. Students solve quadratic equations in Key Concept 3.3: Polynomial and Rational Functions.

Student Task

The task for this part of the lesson is presented on **Handout 1.7.A: Maximizing the Area of an Animal Pen.** Sample student solutions for Part 1 of the task are included in the Facilitating the Task section. Sample student solutions for Part 2 of the task are provided in the Assess and Reflect section of the lesson.

Facilitating the Task

- Give students a few minutes to read through the scenario and Part 1 of Handout 1.7.A. Then allow time for students to ask clarifying questions.
- Once students understand the instructions, let them begin work on Part 1. It is important that each student determine their own three sketches, but students can work collaboratively if you think it would benefit them.

Classroom Ideas

If you have the technology tools available, you could have students input the widths, lengths, and areas of their sketches into a shared spreadsheet. This will reduce the amount of time for collecting data from the class.

Key Concept 4T.1: Radian Measure and Sinusoidal Functions
Lesson 4T.1: Measuring an Angle's Openness

UNIT 4T

Summarizing the Task

- Engage students in a brief discussion about what they notice and what they wonder about the angle they constructed. You can use questions like the following:
  - What do you notice about your angles?
    The angles appear to be congruent.
  - For each of your angles, what is the ratio of the arc length to the radius? Why?
    The ratio of the arc length to the radius is 1 because we constructed an angle that cuts off an arc length of 1 radius.
  - What is the measure of the angle in both degrees and radians?
    The angle measures 1 radian or a little less than 60 degrees.

Meeting Learners' Needs

While comparing angles, students may observe that it is difficult to distinguish between the angles, since there is now more than one line on the circle. This motivates the need to mark the angle and leads students to understand why it is important to construct angles with the same starting position, or initial ray.

Guiding Student Thinking

Students might struggle to reconcile that the angles have congruent measures but different associated arc lengths. This could be because the radian measure of an angle is a ratio of the arc length and the radius of a circle, which differs from other units of measure that are more familiar to students such as inches or degrees. Remind students to think about the "openness" of a central angle in terms of the fraction of the circle it represents.

- In response to the first question, encourage students to hold their stack of papers up to the light to compare the angles. You can connect their observations to the understanding of congruence from geometry that two angles are congruent if one angle can be moved so it coincides exactly with the other without bending or stretching the angle.
- For the last question, some students might use a protractor to determine the angle measure in degrees or estimate that the angle measures about 60 degrees. The goal here is for students to see that they have just created an angle that is 1 radian in measure. This provides an opportunity to define what a radian means for an angle.
- At this point, you can show students a short animation to help students visualize wrapping a radius length around the circle: **preap.org/Desmos-Radian**.

Instructional Rationale: Insight into the strategic design and purpose of the instructional choices, flow, and scaffolding within the model lesson. Rationales often describe how a concept is continued later in the lesson or unit.

Classroom Ideas: Tips related to the logistics of the instruction, such as suggestions for alternative presentation methods, or ways to alleviate pacing concerns.

Meeting Learners' Needs: Optional differentiation strategies to address diverse learning needs, such as ideas for just-in-time skill building during a lesson or ways to break a task into smaller tasks, if needed, to make it more accessible.

Guiding Student Thinking: Ways to facilitate productive student thinking and prevent or address student misconceptions in critical areas of the lesson.

Pre-AP Algebra 2 Assessments for Learning

Pre-AP Algebra 2 assessments function as components of the teaching and learning cycle. Progress is not measured by performance on any single assessment. Rather, Pre-AP Algebra 2 offers a place to practice, to grow, and to recognize that learning takes time. The assessments are updated and refreshed periodically.

LEARNING CHECKPOINTS

Based on the Pre-AP Algebra 2 Course Framework, the learning checkpoints require students to examine data, models, diagrams, and short texts—set in authentic contexts—in order to respond to a targeted set of questions that measure students' application of the key concepts and skills from the unit. All eight learning checkpoints are automatically scored, with results provided through feedback reports that contain explanations of all questions and answers as well as individual and class views for educators. Teachers also have access to assessment summaries on Pre-AP Classroom, which provide more insight into the question sets and targeted learning objectives for each assessment.

The following tables provide a synopsis of key elements of the Pre-AP Algebra 2 learning checkpoints.

Format	Two learning checkpoints per unit Digitally administered with automated scoring and reporting Questions target both concepts and skills from the course framework
Time Allocated	Designed for one 45-minute class period per assessment
Number of Questions	10–12 questions per assessment ▪ 7–9 four-option multiple choice ▪ 3–5 technology-enhanced questions

Domains Assessed	
Learning Objectives	Learning objectives within each key concept from the course framework
Areas of Focus	Three skill categories aligned to the Pre-AP mathematics areas of focus are assessed regularly across all eight learning checkpoints: ▪ greater authenticity of applications and modeling ▪ engagement in mathematical argumentation ▪ connections among multiple representations

Question Styles	Question sets consist of two or three questions that focus on a single stimulus, such as a diagram, graph, or table. Questions embed mathematical concepts in real-world contexts. *Please see page 70 for a sample question set that illustrates the types of questions included in Pre-AP learning checkpoints and the Pre-AP final exam.*

PERFORMANCE TASKS

Each unit includes one performance-based assessment designed to evaluate the depth of student understanding of key concepts and skills that are not easily assessed in a multiple-choice format.

These tasks, developed for high school students across a broad range of readiness levels, are accessible while still providing sufficient challenge and the opportunity to practice the analytical skills that will be required in AP mathematics courses and for college and career readiness. Teachers participating in the official Pre-AP Program will receive access to online learning modules to support them in evaluating student work for each performance task.

Format	One performance task per unit May be administered online or in print ▪ If administered online, then a score report is available. Educator scored using scoring guidelines
Time Allocated	Approximately 45 minutes or as indicated
Number of Questions	An open-response task with multiple parts

Domains Assessed	
Key Concepts	Key concepts and prioritized learning objectives from the course framework
Skills	Three skill categories aligned to the Pre-AP mathematics areas of focus: ▪ greater authenticity of applications and modeling ▪ engagement in mathematical argumentation ▪ connections among multiple representations

PRACTICE PERFORMANCE TASKS

A practice performance task in each unit provides students with the opportunity to practice applying skills and knowledge in a context similar to a performance task, but in a more scaffolded environment. These tasks include strategies for adapting instruction based on student performance and ideas for modifying or extending tasks based on students' needs.

SAMPLE PERFORMANCE TASK AND SCORING GUIDELINES

The following task and set of scoring guidelines are representative of what students and educators will encounter on the performance tasks. (The example below is a practice performance task in Unit 2.)

PRACTICE PERFORMANCE TASK

Using Transformations to Model a Lion's Location

LEARNING OBJECTIVES

2.2.1 Compare a function f with an additive transformation of f, that is, $f(x+k)$ or $f(x)+k$.

2.2.2 Compare a function f with a multiplicative transformation of f, that is, $f(kx)$ or $k \cdot f(x)$.

2.2.3 Construct a representation of the transformation of a function.

PRACTICE PERFORMANCE TASK DESCRIPTION

In this practice performance task, students explore a real-world scenario in which scientists track the distance a lion travels in a day. Students write a function to model the lion's distance from a radio receiver and then use function transformation to write a new function that converts the units of measure for both the independent and dependent variables. This practice performance task provides students with an opportunity to explore additive and multiplicative transformations in a real-world context. Transformations that are applied to the output of a function can be thought of as a composition of functions in which the output of the original function becomes the input of the transformation function. In transformations that are applied to the input of a function, by contrast, the output of the transformation function becomes the input of the original function.

AREAS OF FOCUS

- Greater Authenticity of Applications and Modeling
- Engagement in Mathematical Argumentation
- Connections Among Multiple Representations

SUGGESTED TIMING

~45 minutes

MATERIALS

- calculator (optional)

HANDOUT

- Unit 2 Practice Performance Task: Using Transformations to Model a Lion's Location

AP Connections

This performance task supports AP preparation through alignment to the following AP Calculus Course Skills:

- **2.B** Identify mathematical information from graphical, numerical, analytical, and/or verbal representations.
- **3.E** Provide reasons or rationales for solutions and conclusions.

ELICITING PRIOR KNOWLEDGE

The goal of this task is for students to demonstrate their understanding of using transformations and composition to construct a function that models a contextual scenario.

- To begin, introduce students to the practice performance task. An excerpt from the student handout of the task is shown in the Scoring Student Work section.
- To prepare students to engage in the task, you could ask the following warm-up problems:
  - Consider a general function $y = f(x)$. Compare the transformation $f(x)+1$ to the function $y = f(x)$.

    The transformation $f(x)+1$ is an additive transformation of *f*. It has the effect of adding 1 to the output of *f*. It can also be interpreted as a composition of *f* and $g(x) = x+1$ such that $f(x)+1 = g(f(x))$. The graph of $y = f(x)+1$ looks like the graph of $y = f(x)$ translated vertically up by 1 unit.
  - Compare the transformation $f(x+1)$ to the function $y = f(x)$.

    The transformation $f(x+1)$ is an additive transformation of *f*. It has the effect of adding 1 to each input value before *f* is applied. It can also be interpreted as a composition of *f* and $g(x) = x+1$ such that $f(x+1) = f(g(x))$. The graph of $y = f(x+1)$ looks like the graph of $y = f(x)$ translated horizontally to the left by 1 unit.
  - Compare the transformation $2f(x)$ to the function $y = f(x)$.

    The transformation $2f(x)$ is a multiplicative transformation of *f*. It has the effect of multiplying each output of *f* by 2. It can also be interpreted as a composition of *f* and $g(x) = 2x$ such that $2f(x) = g(f(x))$. The graph of $y = 2f(x)$ looks like the graph of $y = f(x)$ vertically dilated by a factor of 2.
  - Compare the transformation $f(2x)$ to the function $y = f(x)$.

    The transformation $f(2x)$ is a multiplicative transformation of *f*. It has the effect of multiplying each input value by 2 before *f* is applied. It can also be interpreted as a composition of *f* and $g(x) = 2x$ such that $f(2x) = f(g(x))$. The graph of $y = f(2x)$ looks like the graph of $y = f(x)$ dilated horizontally by a factor of $\frac{1}{2}$.

Meeting Learners' Needs
If you find that students need more support to engage with the warm-up problems, you can provide them with a simple, specific function like $f(x) = x^2$ or $f(x) = |x|$.

Meeting Learners' Needs
Students may find it helpful to make a table of values for each function to make sense of the order in which the transformation functions are composed and to keep track of the relationships between the inputs and outputs.

- If students struggle with the warm-up problems, it could indicate that they are not yet fully prepared to engage in the practice performance task. You may find it beneficial to provide a just-in-time review of the concepts critical for success with this task: function composition and transformations of functions.

SUPPORTING STUDENTS

Here are some possible implementation strategies you could use to help support students in engaging with the task.

- **Previewing the Task:** To support students in identifying key features of the problem, you could display the introductory text and graph and read them with the entire class. This particular practice performance task includes a great deal of information that students must read and analyze. Allow some time for students to ask clarifying questions about anything they read or observe, which may decrease the likelihood that they start working on the task with incorrect interpretations of the given information.
- **Collaboration:** To encourage students to engage in academic conversations in mathematics, you could have them work in pairs to complete the task. It is not recommended that students work in groups of three or more. While there is ample work and enough potential discussion areas for two students, some students in groups of more than two may not have an opportunity to engage meaningfully in all parts of the task.
- **Chunking the Task:** To support students who struggle with time management or may be overwhelmed by large tasks, you could chunk the task into several parts. For this task, parts (a) and (b) could be completed together or separately. Parts (c) and (d) would be best completed together, and part (e) could be completed separately from the other parts.
- **Iteration Support:** If you choose to chunk the task, then be sure to spend a few moments discussing the components of the solution with students during each teacher-facilitated check. Focus on what revisions, if any, they could make to their solution to craft a more complete response. Parts (a) and (b) of the task require a written explanation, which students sometimes forget or neglect to include in their response.

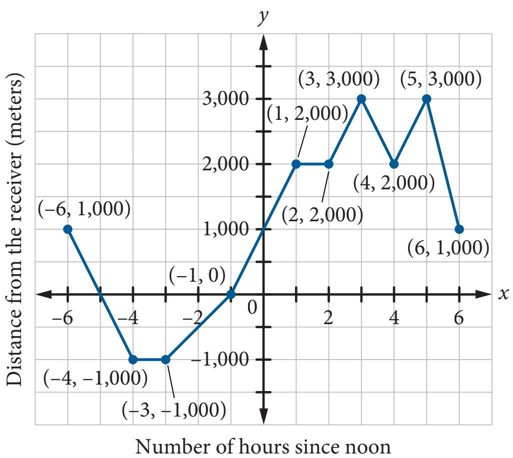
SCORING STUDENT WORK

Whether you decide to have students score their own solutions, have students score their classmates' solutions, or score the solutions yourself, you should use the results of the practice performance task to inform further instruction.

Using Transformations to Model a Lion's Location

African wildlife ecologists study lion populations' behaviors, movements, and interactions in order to develop appropriate conservation plans to better protect these declining populations. This work can be challenging as these large carnivores utilize vast territories across many African countries, such as Botswana, Namibia, Zambia, and Zimbabwe. To monitor the lions' movements, ecologists tag the animals with high-tech collars equipped with radio transmitters that send signals to a stationary receiver, which allows their movements to be tracked and plotted.

One particular ecologist tracks the location of a lion relative to the receiver over the course of one day. From her data, she assigns the number of hours since noon (12 p.m.) as the independent variable and the lion's distance from the receiver (in meters) as the dependent variable. She records the lion's distance from the receiver at different times and constructs a graph of a function model, *f*, shown in the following figure:



Her colleague also constructs a model based on observations of the same lion over the same time period but using different units of measure. So that the two ecologists can compare their models, the first ecologist needs to construct transformations of her function so that the unit of measure for the independent variable is the number of hours since midnight (12 a.m.) and the unit of measure for the dependent variable is kilometers.

There are 12 possible points for this practice performance task.

Student Stimulus and Part (a)

(a) Consider the transformation of the unit of measure for the dependent variable described in the third paragraph. Use the points on the graph of f to write an algebraic representation of a transformation of f such that the dependent variable (the output) of the transformation of f is the distance of the lion from the receiver in kilometers. Explain whether the transformation is an additive or multiplicative transformation. What effect would this transformation have on the graph of f?

Sample Solution	Points Possible
Constructing a transformation of f so that the unit of measure of its dependent variable is kilometers requires dividing the output of f by 1,000. As a result, the output of the transformation yields values in kilometers rather than in meters. The algebraic representation of the transformation of f is $\frac{1}{1{,}000}f(x)$. It is a multiplicative transformation because the transformation involves multiplying the values of the dependent variable by $\frac{1}{1{,}000}$ (or, equivalently, dividing by 1,000). This transformation would have the effect of vertically dilating the graph of f by a factor of $\frac{1}{1{,}000}$. *Scoring note:* Students can receive the third point if they correctly describe the effect on the graph of their incorrect transformation.	**3 points maximum** 1 point for providing a correct algebraic representation of a transformation of f 1 point for providing a correct explanation of why the transformation is a multiplicative transformation 1 point for providing a correct description of the effect the transformation would have on the graph of f

Student Stimulus and Part (b)

(b) Consider the transformation of the unit of measure for the independent variable described in the third paragraph. Use the points given on the graph of f to write an algebraic representation of a transformation of f such that the independent variable (the input) of the transformation is the number of hours since midnight. Explain whether the transformation is an additive or multiplicative transformation. What effect would this transformation have on the graph of f?

Sample Solution	Points Possible
Constructing a transformation of f so that the unit of measure of its independent variable is the number of hours since midnight requires subtracting 12 from the input of the transformation so that it corresponds with the input of f. The algebraic representation of the transformation of f is $f(x-12)$. For example, the ordered pair $(0, 1{,}000)$ for f means that at noon, the lion is 1,000 meters from the receiver. For the transformation of function f, the independent variable should be the number of hours since midnight. Because noon is 12 hours since midnight, an input of 12 should be associated with an output of 1,000 so that the ordered pair (12, 1,000) should be a solution to the transformation of f. Evaluating the function $f(x-12)$ at $x=12$ yields the same output as $x=0$: $f(12-12)=f(0)=1{,}000$. This is an additive transformation because the independent variable is transformed by addition. The transformation would have the effect of horizontally translating the graph of f to the right 12 units. *Scoring note:* Students can receive the third point if they correctly describe the effect on the graph of their incorrect transformation.	**3 points maximum** 1 point for providing a correct algebraic representation of a transformation of f 1 point for providing a correct explanation of why the transformation is an additive transformation 1 point for providing a correct description of the effect the transformation would have on the graph of f

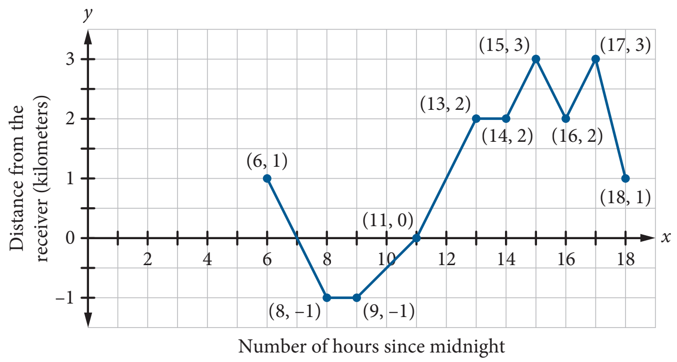
Student Stimulus and Part (c)

(c) Let g be a function that models the relationship between the number of hours since midnight and the lion's distance from the receiver in kilometers. Use your transformations from parts (a) and (b) to write an algebraic representation of g in terms of f.

Sample Solution	Points Possible
An algebraic representation of g that uses the transformations of f from parts (a) and (b) is $g(x)=\frac{1}{1{,}000}f(x-12)$. *Scoring note:* Students can receive this point if they correctly combine the transformations they wrote in parts (a) and (b), even if one or both of those transformations are incorrect.	**1 point maximum** 1 point for combining the transformations of f from parts (a) and (b) to form function g

Student Stimulus and Part (d)

(d) Sketch a graph of the function g that you constructed in part (c) using the coordinates given on the graph of f.

Sample Solution	Points Possible
 A graph of g is shown in the figure. *Scoring note:* Students can receive full credit for this part if their graph for function g corresponds to their algebraic representation of g from part (c), even if the algebraic representation is incorrect.	**2 points maximum** 1 point for representing the horizontal translation 1 point for representing the vertical dilation

Student Stimulus and Part (e)

(e) Identify the intercepts and interpret their meaning in the graph of g. How does this meaning compare to the meaning of the x-intercepts of the graph of f?

Sample Solution	Points Possible
The x-intercepts of the graph of g are $(7,\ 0)$ and $(11,\ 0)$. This means that at 7 hours since midnight, or 7 a.m., and at 11 hours since midnight, or 11 a.m., the lion is 0 kilometers from the receiver. The values of the x-intercepts are different for the graph of f, but the meanings are the same. The x-intercepts for the graph of f are $(-5,\ 0)$ and $(-1,\ 0)$. These correspond to 5 hours before noon, or 7 a.m., and one hour before noon, or 11 a.m. The lion is 0 meters from the receiver at these times.	**3 points maximum** 1 point for identifying the x-intercepts of the graph of g 1 point for providing a correct interpretation of the meaning of the x-intercepts of g with correct units 1 point for providing a correct comparison of the x-intercepts for the graphs of f and g

PROVIDING FEEDBACK ON STUDENT WORK

After scoring your students' work, it is important to identify trends in their responses to inform further instruction. These trends should include topics that students consistently displayed mastery of, as well as conceptual errors that students commonly made. Possible trends and suggested guidance for each part of the task follow, although the patterns you observe in your classroom may differ.

(a) If students have difficulty identifying whether they should multiply or divide by 1,000 to convert from meters to kilometers, it could be helpful to have them make a table of values with equivalent measurements in both units to look for a pattern. Also, students may find a graphing utility to be a useful tool for identifying the effect a multiplicative transformation has on the graph of a function.

Teacher Notes and Reflections

(b) If students have difficulty identifying whether they should add or subtract 12 to the input of the function transformation, it may be helpful to have them make a table of values with different times of day represented using both units of measure. Also, students may find a graphing utility to be a useful tool for identifying the effect of an additive transformation on the graph of a function.

Teacher Notes and Reflections

(c) Students may have difficulty using function composition to combine the two transformed functions into a single transformed function model. They may benefit from creating a table of values with columns for hours since midnight, hours since noon, distance in meters, and distance in kilometers to see how the desired input (number of hours since midnight) can be associated with the desired output (distance in kilometers) using the given data.

Teacher Notes and Reflections

(d) If students cannot produce a correct graph by analyzing the graph of f and their algebraic representation of the function g, they may benefit from using a dynamic graphing utility to look for patterns in the effect that different transformations have on the graph of function f.

Teacher Notes and Reflections

(e) Some students could experience difficulty interpreting the meaning of the x-intercepts of the graphs because they do not see the connections between the two representations. These students may benefit from some additional practice interpreting a graph generated from a different contextual scenario.

Teacher Notes and Reflections

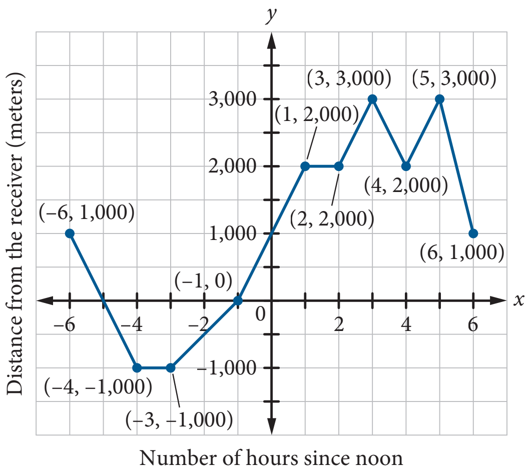
Assure students that converting their score into a percentage does not provide an accurate measure of how they performed on the task. You can use the following suggested score interpretations with students to discuss their performance.

Points Received	How Students Should Interpret Their Score
11 or 12 points	"I know all of these algebraic concepts really well. This is top-level work. (A)"
8 to 10 points	"I know all of these algebraic concepts well, but I made a few mistakes. This is above-average work. (B)"
5 to 7 points	"I know some of these algebraic concepts well, but not all of them. This is average-level work. (C)"
2 to 4 points	"I know only a little bit about these algebraic concepts. This is below-average work. (D)"
0 or 1 point	"I don't know much about these algebraic concepts at all. This is not passing work. (F)"

Using Transformations to Model a Lion's Location

African wildlife ecologists study lion populations' behaviors, movements, and interactions in order to develop appropriate conservation plans to better protect these declining populations. This work can be challenging as these large carnivores utilize vast territories across many African countries, such as Botswana, Namibia, Zambia, and Zimbabwe. To monitor the lions' movements, ecologists tag the animals with high-tech collars equipped with radio transmitters that send signals to a stationary receiver, which allows their movements to be tracked and plotted.

One particular ecologist tracks the location of a lion relative to the receiver over the course of one day. From her data, she assigns the number of hours since noon (12 p.m.) as the independent variable and the lion's distance from the receiver (in meters) as the dependent variable. She records the lion's distance from the receiver at different times and constructs a graph of a function model, *f*, shown in the following figure:



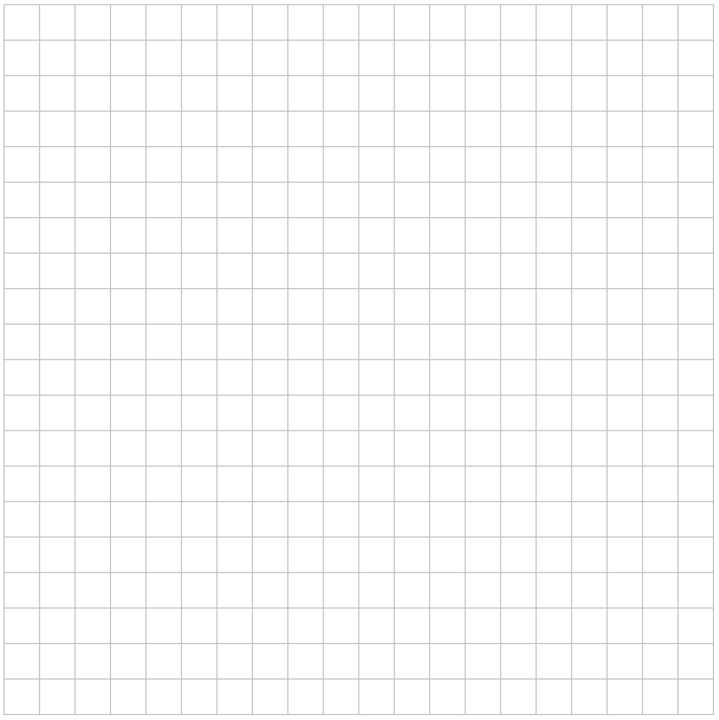
Her colleague also constructs a model based on observations of the same lion over the same time period but using different units of measure. So that the two ecologists can compare their models, the first ecologist needs to construct transformations of her function so that the unit of measure for the independent variable is the number of hours since midnight (12 a.m.) and the unit of measure for the dependent variable is kilometers.

(a) Consider the transformation of the unit of measure for the dependent variable described in the third paragraph. Use the points on the graph of f to write an algebraic representation of a transformation of f such that the dependent variable (the output) of the transformation of f is the distance of the lion from the receiver in kilometers. Explain whether the transformation is an additive or multiplicative transformation. What effect would this transformation have on the graph of f?

(b) Consider the transformation of the unit of measure for the independent variable described in the third paragraph. Use the points given on the graph of f to write an algebraic representation of a transformation of f such that the independent variable (the input) of the transformation is the number of hours since midnight. Explain whether the transformation is an additive or multiplicative transformation. What effect would this transformation have on the graph of f?

(c) Let g be a function that models the relationship between the number of hours since midnight and the lion's distance from the receiver in kilometers. Use your transformations from parts (a) and (b) to write an algebraic representation of g in terms of f.

(d) Sketch a graph of the function g that you constructed in part (c) using the coordinates given on the graph of f.



(e) Identify the intercepts and interpret their meaning in the graph of g. How does this meaning compare to the meaning of the x-intercepts of the graph of f?

FINAL EXAM

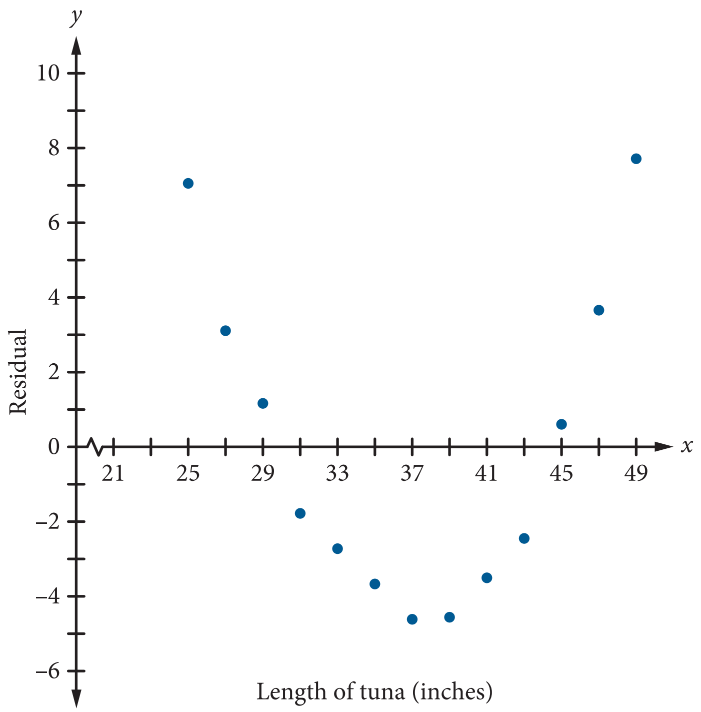
Starting in the school year of 2023–24, Pre-AP Algebra 2 will include a final exam featuring multiple-choice and technology-enhanced questions as well as an open-response question. The final exam will be a summative assessment designed to measure students' success in learning and applying the knowledge and skills articulated in the Pre-AP Algebra 2 Course Framework. The final exam's development will follow best practices such as multiple levels of review by educators and experts in the field for content accuracy, fairness, and sensitivity. The questions on the final exam will be pretested, and the resulting data will be collected and analyzed to ensure that the final exam is fair and represents an appropriate range of the knowledge and skills of the course.

The final exam will be delivered on a secure digital platform in a classroom setting. Educators will have the option of administering the final exam in a single extended session or in two shorter consecutive sessions to accommodate a range of final exam schedules.

Multiple-choice and technology-enhanced questions will be delivered digitally and scored automatically with detailed score reports available to educators. This portion of the final exam will build on the question styles and formats of the learning checkpoints; thus, in addition to their formative purpose, the learning checkpoints provide practice and familiarity with the final exam. The open-response question, modeled after the performance tasks, will be delivered as part of the digital final exam but scored locally by educators.

SAMPLE ASSESSMENT QUESTIONS

The following questions are representative of what students and educators will encounter on the learning checkpoints and final exam.



A particular fishing company records the length, in inches, and the weight, in pounds, of the yellowfin tuna that they catch. To determine if there is a linear relationship between the length and weight, the company constructs a regression equation, whose residual plot is shown in the figure. Which of the following conclusions about the relationship between the length and weight of the tuna is best supported using the information presented in the residual plot?

(A) A linear model is appropriate for the relationship between the length and weight because the residual plot displays a quadratic pattern.

(B) A linear model is appropriate for the relationship between the length and weight because the residual plot displays no observable pattern.

(C) A linear model is not appropriate for the relationship between the length and weight because the residual plot displays a pattern.

(D) A linear model is not appropriate for the relationship between the length and weight because the residual plot displays no observable pattern.

Assessment Focus

In problem 1, students evaluate the appropriateness of using a linear equation to model the relationship between two variables by determining if the associated residual plot for the function model displays a discernable pattern.

Correct Answer: C

Learning Objective:

1.1.2 Use residual plots to determine whether a function model appropriately models a data set.

Area of Focus: Engagement in Mathematical Argumentation

The function h is defined as $h(x)=\sqrt{x^2+1}$. If h can be formed by the composition of functions f and g such that $h(x) = f(g(x))$, then which TWO of the following pairs of functions could be f and g?

(A) $f(x)=x^2+1$ and $g(x)=\sqrt{x}$

(B) $f(x)=\sqrt{x+1}$ and $g(x)=x^2$

(C) $f(x)=x^2$ and $g(x)=\sqrt{x+1}$

(D) $f(x)=\sqrt{x-1}$ and $g(x)=x^2+2$

Assessment Focus

In problem 2, students decompose a composite function into two individual functions. The multiple responses required to correctly answer the question reinforces for students that there are multiple correct ways to decompose a function.

Correct Answer: B and D

Learning Objective:

2.1.3 Express a given algebraic representation of a function in an equivalent form as the composition of two or more functions.

Area of Focus: Connections Among Multiple Representations

Let the functions f and g be defined as follows: $f(x) = \log_b(x)$ and $g(x) = \log_b(x - c)$, where c is a positive real number. Which of the following statements correctly describes the relationship between the graphs of $y = f(x)$ and $y = g(x)$?

(A) The graph of $y = g(x)$ coincides with the graph of $y = f(x)$ for all values of $x > c$.

(B) The graph of $y = g(x)$ is a dilation of the graph of $y = f(x)$ by a factor of c.

(C) The graph of $y = g(x)$ is a vertical translation of the graph of $y = f(x)$ down c units.

(D) The graph of $y = g(x)$ is a horizontal translation of the graph of $y = f(x)$ to the right c units.

Assessment Focus

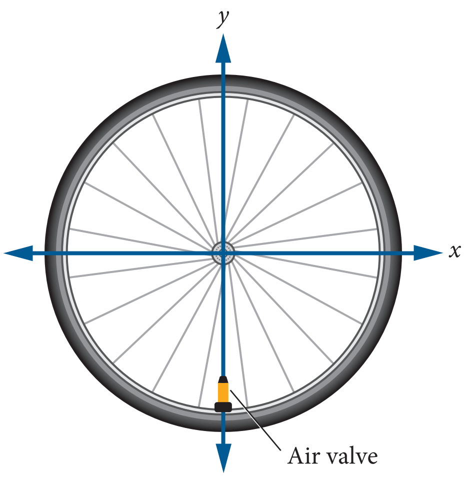
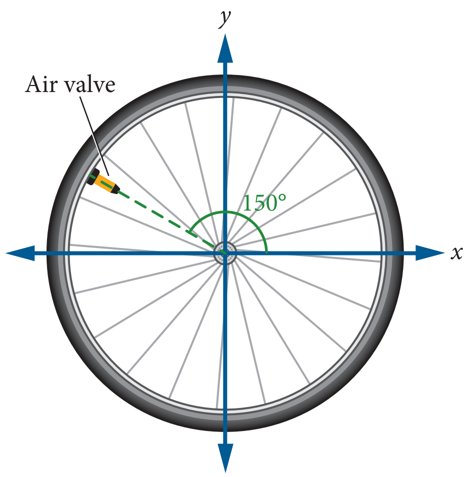
In problem 3, students compare the algebraic forms of two related logarithmic functions to determine the relationship between their graphs.

Correct Answer: D

Learning Objective:

3.1.3 Construct a representation of a logarithmic function.

Area of Focus: Connections Among Multiple Representations

Starting Position	**Final Position**
	

A bicycle tire with a diameter of 700 millimeters is shown at its starting and final positions. At the start of a bicycle ride, the front tire's air valve is positioned at the bottom of the tire. After riding the bicycle, the final position of the air valve forms an angle in standard position, relative to the horizontal axis of the bicycle tire, that measures 150°. Which of the following expressions represents the height of the air valve from the ground?

(A) $350\cos\left(\frac{5\pi}{6}\right)$

(B) $350 + 350\cos\left(\frac{5\pi}{6}\right)$

(C) $350\sin\left(\frac{5\pi}{6}\right)$

(D) $350 + 350\sin\left(\frac{5\pi}{6}\right)$

Assessment Focus

In problem 4T, students are expected to apply their knowledge of the relationship between the coordinates of a point on a circle and a related central angle to answer a question in a real-world context.

Correct Answer: D

Learning Objective:

4T.1.4 Determine the exact coordinates of any point on a circle centered at the origin.

Area of Focus: Greater Authenticity of Applications and Modeling

$$T(x, y) = \begin{bmatrix} -2 & 1 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

In the coordinate plane, the transformation T maps point A to a point with coordinates (–10, 20). Which of the following ordered pairs are the coordinates of point A?

(A) $(-25, -15)$

(B) $(5, 0)$

(C) $(40, -100)$

(D) $(100, -70)$

Assessment Focus

In problem 4M, students are expected to multiply the column matrix that represents the image point, $\begin{bmatrix} -10 \\ 20 \end{bmatrix}$, by the inverse of the transformation matrix, $\begin{bmatrix} -1.5 & -0.5 \\ -2 & -1 \end{bmatrix}$, to determine the matrix form of the preimage point A, $\begin{bmatrix} 5 \\ 0 \end{bmatrix}$.

Correct Answer: B

Learning Objective:

4M.1.8 Determine the preimage of a specified point under a given linear transformation.

Area of Focus: Connections Among Multiple Representations

Pre-AP Algebra 2 Course Designation

Schools can earn an official Pre-AP Algebra 2 Course Designation by meeting the Pre-AP Program commitments summarized below. Pre-AP Course Audit Administrators and teachers will complete a Pre-AP Course Audit process to attest to these commitments. All schools offering courses that have received a Pre-AP Course Designation will be listed in the Pre-AP Course Ledger, in a process similar to that used for listing authorized AP courses.

PROGRAM COMMITMENTS

- Teachers have read the most recent *Pre-AP Algebra 2 Course Guide*.
- The school ensures that Pre-AP course frameworks and assessments serve as the foundation for all sections of the course at the school. This means that the school must not establish any barriers (e.g., test scores, grades in prior coursework, teacher or counselor recommendation) to student access and participation in the Pre-AP Algebra 2 coursework.
- Teachers administer at least one of two learning checkpoints per unit on Pre-AP Classroom and one performance task per unit.
- Teachers complete the foundational professional learning (Online Foundational Modules or Pre-AP Summer Institute) and at least one online performance task scoring module. The current Pre-AP coordinator completes the Pre-AP Coordinator Online Module.
- Teachers align instruction to the Pre-AP Algebra 2 Course Framework and ensure their course meets the curricular commitments summarized below.
- The school ensures that the resource commitments summarized below are met.
- Please note if a state's standards do not include trigonometry in Algebra 2, then the school may fulfill the Pre-AP course commitments with Unit 4M rather than Unit 4T. Schools have to choose only one fourth unit.

CURRICULAR COMMITMENTS

- The course provides opportunities for students to develop an understanding of the Pre-AP Algebra 2 key concepts and skills articulated in the course framework through the four units of study.
- The course provides opportunities for students to engage in the Pre-AP shared instructional principles.
  - close observation and analysis
  - evidence-based writing

  - higher-order questioning
  - academic conversation
- The course provides opportunities for students to engage in the three Pre-AP mathematics areas of focus. The areas of focus are:
  - greater authenticity of applications and modeling
  - engagement in mathematical argumentation
  - connections among multiple representations
- The instructional plan for the course includes opportunities for students to continue to practice and develop disciplinary skills.
- The instructional plan reflects time and instructional methods for engaging students in reflection and feedback based on their progress.
- The instructional plan reflects making responsive adjustments to instruction based on student performance.

RESOURCE REQUIREMENTS

- The school ensures that participating teachers and students are provided computer and internet access.
- Teachers should have consistent access to a video projector for sharing web-based instructional content and short web videos.

Accessing the Digital Materials

Pre-AP Classroom is the online application through which teachers and students can access Pre-AP instructional resources and assessments. The digital platform is similar to AP Classroom, the online system used for AP courses.

Pre-AP coordinators receive access to Pre-AP Classroom via an access code delivered after orders are processed. Teachers receive access after the Pre-AP Course Audit process has been completed.

Once teachers have created course sections, students can enroll in them via a join code. When both teachers and students have access, teachers can share instructional resources with students, assign and score assessments, and complete online learning modules; students can view resources shared by the teacher, take assessments, and receive feedback reports to understand progress and growth.